THE JOURNAL OF THE LEARNING SCIENCES, *11*(2&3), 217–249

Teacher's and Students' Understanding: The Role of Tools and Inscriptions in Supporting Effective Communication

Kay McClain
Department of Teaching and Learning
Vanderbilt University

The 2 episodes featured in this issue provide a rich setting in which to investigate not only the influence but also the confluence of students' and teacher's understandings on the quality of whole class discussions. In particular, I focus on the communication between the students and myself as the teacher in the classroom. This unique perspective allows me to offer insights into the teacher's decision-making process and how those decisions influenced the opportunities for learning. As part of the analysis I consider the role of tools in both supporting and constraining communication in the classroom. The analysis therefore makes explicit the tensions in teaching by highlighting the importance of the teacher's understanding of the students' offered explanations and justifications and the mathematics that is to be taught.

> Students' actions and constructions are influenced by a teacher's actions and constructions, and vice versa. But neither is totally dependent on or independent of the other.

Research on effective teaching often characterizes the teacher's classroom decision-making process as informed by the mathematical agenda but constantly being revised and modified in action based on students' contributions (cf. Ball, 1990, 1993; Carpenter & Fennema, 1991; Cobb, Yackel, & Wood, 1991; Lampert, 1990; Maher, 1987; Simon, 1997; Simon & Schifter, 1991; A. C. Thompson, 1992). In particular, Ball (1993) described her teaching as keeping an ear on the students and an "eye on the mathematical horizon" (p. 185). Lampert used the metaphor of a dance instructor who "dances with a learner so that the learner will know what it

Correspondence and requests for reprints should be sent to Kay McClain, 5330 Old Haring Road, Franklin, TN 37064. E-mail: mcclain@vanderbilt.edu

feels like to be interacting with someone who knows how to do what he or she is trying to learn how to do" (p. 42). Elsewhere, my colleague and I (McClain & Cobb, 2001) have described teaching as "pursuing a potentially revisable agenda as informed by inferences about students' beliefs and understandings" (p. 237). Each of these characterizations takes account of the students' contributions while attending to mathematics. Attempting to balance the tension inherent in simultaneously attending to students' offered solutions and the mathematical agenda is the hallmark of deliberately facilitated discussions (cf. McClain, in press). These discussions involve a plethora of decisions that must be made both before and while interacting with students. The image that results is that of the teacher constantly judging the nature and quality of the students' contributions against the mathematical agenda to ensure that the issues under discussion offer means of supporting students' development. This view of mathematical discussions stands in stark contrast to open-ended sessions where all students are allowed to share their solutions without concern for potential mathematical contributions.

A critical resource for the teacher in this process is therefore the means of support available to help her achieve her mathematical agenda. This support manifests itself in the form of the instructional tasks and the tools available for solving the tasks. For this reason tools, notation systems, and student-generated inscriptions all serve an important role in the mathematics classroom. However, it is not the tool (or the notation or the inscription) in isolation that offers support for the teacher. It is instead the students' use of the tools and the meanings that they come to have as a result of this activity (Kaput, 1994; Meira, 1995, 1998; van Oers, 1996). In this way, the tool is not seen as standing apart from the activity of the student.

The two episodes featured in this volume provide a rich setting in which to investigate not only the influence but also the confluence of students' and teacher's understandings of tools and inscriptions on the quality of whole-class discussions. In particular, I will focus my analysis on the two computer-based tools for analysis and the students' informal inscriptions and how they became constituted in the course of classroom discussions. In doing so I will focus on communication between the students and myself as the teacher in the classroom, and how the tools and inscriptions both supported and constrained that communication. The analysis will therefore make explicit the tensions in teaching by highlighting the importance of the teacher's understanding of the students' offered explanations and justifications and the mathematics that is to be taught.

When focusing on students' offered explanations and justifications, the teacher is seen to actively guide the mathematical development of both the classroom community and individual students (Ball, 1993; Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Wood, & Yackel, 1993; Lampert, 1990). This guiding necessarily requires a sense of *knowing in action* on the part of the teacher as he or she attempts to capitalize on opportunities for mathematical learning that emerge from students' activity and explanations. With this comes the responsibility of monitoring classroom discussions, engaging in productive mathematical discourse, and providing direction and guidance as judged appropriate. Similar pedagogical issues were addressed by Lampert in her discussion of the teacher's role in guiding mathematical argumentation as a "zigzag" between conjectures and refutations. This characterization of the role of the teacher is also evident in Simon's (1995) account of the "mathematics teaching cycle," which highlights the relation between teachers' knowledge, their goals for students, and their interaction with students. Teachers who approach teaching as a generative problem-solving activity—in the course of which they modify their goals, their understandings of students' thinking and of mathematics, and the means they use to support students' mathematical development—contribute to the improvement of classroom practice (cf. Ball, 1990; Carpenter & Fennema, 1991; Cobb et al., 1991; Franke, Carpenter, Levi, & Fennema, 1998; Maher, 1987; Shulman, 1986; Simon & Schifter, 1991; A. G. Thompson, 1992). A focus on the importance of students' contributions also highlights the importance of norms that constitute the classroom participation structure (Erickson, 1986; Hershkowitz & Schwartz, 1999; Lampert; Sfard, 2000; Simon & Blume, 1996; Voigt, 1995; Yackel & Cobb, 1996). The importance attributed to classroom norms stems from the contention that students reorganize their specifically mathematical beliefs and values as they participate in and contribute to the establishment of these norms (cf. Bowers & Nickerson, 1998; Lampert; Simon & Blume; Voigt).

The importance of a strong knowledge of content in teaching mathematics has been acknowledged by a variety of scholars (Ball, 1989, 1993, 1997; Bransford, Brown, & Cocking, 2000; Grossman, 1990; Grossman, Wilson, & Schulman, 1989; Ma, 1999; Morse, 2000; National Research Council, 2001; Schifter, 1995; Shulman, 1986; Sowder, Philipp, Armstrong, & Schappelle, 1998; Stein, Baxter, & Leinhardt, 1990). In doing so they emphasized the importance of a deep understanding of the mathematics one will teach so that decisions made in action about appropriate avenues of exploration can be framed against a relatively sophisticated understanding of the mathematical concepts to be explored. Teachers therefore need to have a strong sense of what constitutes the mathematical basis of each investigation that they conduct in their classroom. The work of Carpenter and his colleagues in cognitively guided instruction (Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1996; Fennema, Carpenter, Franke, & Carey, 1993) points to the importance of teachers understanding not only the mathematics that they teach but also their students' mathematics. It is only in understanding their students' ways of reasoning that teachers can develop instruction that supports their students' development. This approach to instruction can be contrasted with an enactment model where curricular materials are introduced in a lock-step manner as outlined by the developer. This latter form of instruction assumes that the materials attend to the students' diverse ways of reasoning and varied levels of understanding. Instead, teachers who attend to students take account of their activity on a daily and ongoing basis as they modify their instructional plan. To teach in this manner, teachers have to find ways to make students' learning visible in the course of ongoing classroom discussions and assignments. To benefit from these varied assessments, teachers need knowledge of the range and variety of solutions and strategies students might use in task situations. Simon (1996) supported this premise by noting that "it stands to reason that instruction that is sensitive to students' knowledge and their processes of learning is likely to be more effective" (p. 37).

In the following sections of this article, I first describe the methodology that guided the retrospective analysis of the data from the teaching experiment.[1] Against this background I analyze the two classroom episodes. My analysis is grounded in both a documentation of my decision-making process as recorded in my daily journal and a retrospective analysis of these decisions.[2] In the course of my analysis I highlight the role of tools and inscriptions in both enabling and constraining my interactions with the students.

METHOD

Theoretical Framework

The emergent perspective is the theoretical framework that guided the detailed longitudinal analyses of the classroom teaching experiment (see Figure 1). It explicitly coordinates sociological and psychological perspectives. The psychological perspective is constructivist and treats mathematical development as a process of self-organization in which the learner reorganizes his or her activity in an attempt to achieve purposes or goals (von Glaserfeld, 1995). The sociological perspective is interactionist and views communication as a process of mutual adaptation wherein individuals negotiate mathematical meaning (Bauersfeld, Krummheuer, & Voigt, 1988). From this latter perspective, learning is characterized as the personal reconstruction of societal means and models through negotiation in interaction. Together, the two perspectives treat mathematical learning as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society. Individual and collective processes are viewed as reflexively

[1]The research team members involved in the teaching experiment included myself, Paul Cobb, Koeno Gravemeijer, Maggie McGatha, Lynn Hodge, Jose Cortina, Carla Richards, and Cliff Konold.

[2]Although I had primary responsibility for the teaching during the classroom teaching experiment, on numerous occasions a second member of the research team participated in whole class discussions. In addition, additional members of the research team occasionally circulated around the room to monitor students' activity during small-group work.

Social	Psychological
Classroom Social Norms	Beliefs about own role, others' roles, and the general nature of mathematical activity
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical norms	Mathematical interpretations and reasoning.

FIGURE 1 The emergent perspective.

related in that one does not exist without the other (Cobb, Perlwitz, & Underwood, 1994). The arrows in Figure 1 are intended to indicate this reflexive relationship.

This coordination of the social and psychological perspectives within the emergent perspective is highlighted by the two columns shown in Figure 1 (cf. Cobb & Yackel, 1996). The entries in the column headed "Social" indicate three aspects of the classroom microculture that we have found useful to differentiate when analyzing the classroom community. The three corresponding entries in the column headed "Psychological" indicate three related aspects of individual students' reasoning as they participate in communal classroom processes.

For the purposes of this article, I focus on the aspect of the framework that deals with sociomathematical norms (cf. McClain & Cobb, 2001; Yackel & Cobb, 1996). Sociomathematical norms include what counts as a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation and justification (cf. Hershkowitz & Schwartz, 1999; Lampert, 1990; Simon & Blume, 1996; Voigt, 1995). In contrast, social norms can be characterized as general norms that are necessary for engaging in classroom discussions and can apply to any subject matter. Such norms include explaining and justifying solutions, attempting to make sense of explanations given by others, and challenging others' thinking (Cobb & Yackel, 1996). As indicated in the figure, the psychological correlates of the sociomathematical norms are taken to be the specifically mathematical beliefs and values that constitute what might be termed students' *mathematical dispositions*. In this perspective, students are seen to reorganize their understandings of what counts as different, sophisticated, efficient, or acceptable solutions and explanations as they participate in the negotiation of sociomathematical norms.

The analyses presented in this article focus on the evolution of the sociomathematical norm of what counts as an acceptable mathematical argument, the norms for justification for these arguments, and the role of tools in both enabling and constraining these arguments. However, it is important to note that a focus on the sociomathematical norm of what counts as an acceptable mathematical argument makes assumptions about the establishment of certain social norms. It is therefore important that both the social and sociomathematical norms were a focus of attention by the research team throughout the teaching experiment.

Method of Analysis

The approach I took when conducting retrospective analyses of the data generated during the teaching experiment involves a method described by Cobb and Whitenack (1996) for analyzing large sets of classroom data. This method is an adaptation of Glaser and Strauss's (1967) constant comparative method. The initial orientation for a retrospective analysis is provided by the tentative and eminently revisable conjectures that the research team developed while actually conducting the teaching experiment. The method involves continually testing and revising conjectures while working through the data chronologically episode by episode. This constant comparison of conjectures with data results in the formulation of claims or assertions that span the entire data set but yet remain empirically grounded in the details of specific episodes. For this reason, the analysis of the two episodes featured in this issue can be used to characterize the general ways of participating in the classroom that were normative at these particular periods of time during the teaching experiment.

ANALYSIS OF CLASSROOM EPISODES

Before I present the analysis of the classroom episodes, it is important to provide background information on the processes used by the research team before entering the classroom and its importance to me as the teacher. In particular, in the months before the classroom teaching experiment, the research team mapped out a hypothetical learning trajectory that guided the development of the instructional sequence. It was a conjecture about the learning route of the classroom community and was constantly subject to change and modification based on ongoing analysis of the classroom sessions. Integral to the trajectory were conjectures about how to support this development. This included design decisions about tasks and the development of tools for analysis. As a result, the two computer-based tools were designed to offer resources that would support emergence of the conjectured learning trajectory.

It is important to clarify that this conjecture about a learning route and the means of supporting it guided my decision-making process as I interacted with students daily in the classroom. For that reason, I was continually attempting to gauge the students' current understandings against the mathematical agenda. It was therefore critical that I keep the envisioned mathematical endpoint as my guide. At the end of each class session, the research team discussed the day's events and made any necessary revisions to the instructional sequence based on observations of the students' activity. This required not only that I have a deep understanding of the mathematical purpose of each classroom task, but that I also understand students' solutions so as to be able to make reasoned decisions about how best to proceed. In doing so, I had to take account of tools and other resources available to support this goal.

Batteries Episode

As will become apparent from my analysis of the *Batteries* episode, I initially found it very difficult to balance these multiple agendas. As a result, I was unable to communicate effectively with the students during discussions. My understanding of their understanding was confounded by my interpretation of their use of the computer minitool. Because of my preconceived judgments and anticipations about solutions for the task and efficient tool use, I had difficulty adjusting to theirs. I was attempting to only influence instead of also be influenced by the students' actions and constructions (cf. Steffe & Thompson, 2000). For this reason, my preconceptions about efficient tool use constrained my ability to capitalize on the students' activity with the tool.

Classroom activity structure. Before I proceed with the analysis, it is important to clarify the classroom activity structure. This structure was composed of three general phases, often spanning more than one class period. The first of these involved the introduction of the task. Because the research team decided to use archival data in the teaching experiment, we judged that it would be important for the students and myself to talk through what the research team came to call the data creation process (cf. Tzou, 2000). In effect, we wanted the students to elaborate the design specifications that would yield the data they were to analyze. This aspect of the classroom typically was characterized by students clarifying for themselves the question to be answered or dilemma to be resolved and the ramifications of the data collection procedures on their analysis. During these discussions, the students would create a list of attributes on which they would want to generate measures to answer the question. This was followed by discussions of how to collect the data, what measures would be appropriate, and how to account for differences in data collection procedures. This process proved important in grounding the students' activity in the context of a situation that had real consequences.

My initial interpretation of this phase of the classroom activity structure entailed my telling the students a story about a situation that ultimately would require the needed analysis. In these instances, the students often recounted incidents from their personal experience but did not actually engage in the process of data creation. After these discussions, I presented the data the students were to analyze and attempted to support their understanding of the process by which the data were generated by offering an explanation. Because they had not engaged in working through the data creation process, their analyses often lacked grounding in the context of the question at hand.

As an example, in introducing the batteries task, I asked the students if they were familiar with different brands of batteries. I followed by asking them if they knew what most battery companies "claimed." After some suggestions, one student noted that most of them claim to last the longest. Another student stated that some of them had to be lying if they all claimed to last the longest. There was an extensive discussion about this and students even talked about the "small print" that companies use when they explain how the batteries were tested. I then asked the students to make suggestions as to how we might test batteries to determine their longevity. They responded by suggesting that the different brands of batteries be placed in similar devices and timed to see how long the device would run.

Although I engaged the students in a conversation related to batteries and how we might test them, the students did not discuss what data would be needed to make a good decision (e.g., data on the time that the batteries lasted). I imposed those criteria. As a result, their framing of the task was not grounded in a process of data creation that would generate measures to be used in answering the question at hand. They only responded to my questions about how to test for longevity. This in turn contributed to some of the students' subsequent inability to situate their analysis in the context of the investigation.

The second phase of the classroom activity structure involved the students working in pairs or small groups at the computers to analyze the data. During this phase, some of my colleagues and I would circulate among the groups of students to monitor their work. It is important to clarify that during this time my role was not to intervene and correct students' errors or to suggest strategies for the particular task. It was, instead, to gain an understanding of the varied and diverse ways students were approaching the task so that I could plan for the subsequent whole class discussion. To this end, I would identify students whose solutions, when discussed, could support the emergence of the mathematical agenda. This might entail comparing and contrasting two or more solutions or discussing a sequence of solutions intended to build toward a mathematical endpoint.

The third and final phase of the classroom activity structure entailed a discussion and critique of the students' analysis. During this phase, my role was to deliberately facilitate the discussion by selecting students to share their solutions and highlighting aspects of those solutions that were mathematically significant. Simultaneously, I was attending to the negotiation of norms for argumentation. This highlights the importance of the relation between the negotiation of classroom social and sociomathematical norms and the students' mathematical development.

Each of the three phases of the classroom activity structure depended on effective communication between the students and myself. As will become apparent from my analysis of the *Batteries* episode, my understandings and interpretations of the students' explanations and tool use often interfered with this communication.[3] I was unable to adjust my interpretations to accommodate the students' ways of reasoning with the tool. For this reason, the discussions failed to provide an effective venue for productive mathematical discussions.

Instructional intent of the batteries task. Before introducing the batteries data, I worked to understand the instructional intent of the task. This was necessary to guide my decision-making process as I interacted with the students. This not only involved an understanding of how the task fit in the instructional sequence as outlined by the conjectured learning trajectory, but it also involved making anticipations about how students might reason on the task and subsequently structure the data as they worked with the computer minitool. Because this was only the second task presented using the first minitool, a primary goal included supporting students' ability to act on the bars in the minitool as data. The research team wanted to ensure that the students' activity take on the characteristics of genuine data analysis instead of simply manipulating numbers in an attempt to complete an assignment. For this reason, situations were chosen in the initial investigations such that the attribute measured had a sense of linearity (e.g., braking distances of cars and length of time a battery lasted). We conjectured that the bars would therefore be a linear representation of the measure.

A second aspect of this goal was that the students would find ways to characterize the data that attended to the question. This was based on our analysis of preassessment data which indicated that the students typically calculated the mean when comparing two sets of data. Our goal was to create a perturbation in this way of thinking and initiate shifts toward ways of reasoning that focused on features of the data sets including variability. This was supported by features on the computer

[3]These interpretations were based on analysis of my daily reflective journal.

minitool that allowed the students to order and partition the data. The goal was then that the students would come to reason about the characteristics of the data values as they made comparisons across the data sets. The research team also had an overarching goal of supporting students' ability to reason multiplicatively about data (cf. Harel & Confrey, 1994; P. W. Thompson, 1994; Thompson & Saldanha, 2000). Multiplicative reasoning is distinguished from reasoning additively. As an illustration, in reasoning additively, students might focus on the absolute frequency of the data in particular intervals (e.g., 15 cars are going faster than the speed limit and 30 are going slower). In contrast, in reasoning multiplicatively, students might focus on the relative frequency or proportion of the data (e.g., one third of the cars are going faster than the speed limit). As a result of our noting the importance of this distinction, I had anticipated that students might partition the data sets using the tool and reason about proportions of the data that fell within or above a certain range or cut point. These would be solutions that I would want to capitalize in the course of whole class discussion.

The features on the computer minitool were designed to support this type of reasoning. In particular, the range tool was designed to allow students to capture a subset of the data values to make decisions about the range or variability. The value bar was intended to offer students a means of partitioning the data, identifying the median, or estimating the mean. Our intent was that the students would use these features of the minitool as they reasoned about how to structure the data to make a judgment or comparison. In this way, the tool offered a means of supporting the mathematical agenda.

Classroom analysis. As the students worked at the computers on their analyses, I circulated among the groups to identify students whose ways of reasoning would support the emergence of the aforementioned goals. In doing so, I took a "snapshot" of their activity and attempted to determine how they were proceeding with their analysis as I made decisions about how to organize the whole class discussion. From my brief exchange with Ceasara and her partner, I inferred that they had partitioned the data set with the range tool feature on the computer minitool and reasoned about proportions of each brand of battery on either side of the cut point. I judged this to be an efficient and sophisticated way to approach this task and one that supported my mathematical goals. I therefore selected Ceasara to share in hopes of highlighting this solution method in the whole class discussion. My judgment about their solution was based, in part, on the way Ceasara and her partner had organized the data with the computer minitool. They had used the range tool to identify the top 10 values as shown in Figure 2 (see also Figure 4 in the Appendix, article by McClain, this issue).

I interpreted their use of the range tool as a way to partition the data. In doing so, I assumed that Ceasara and her partner had reasoned about proportions of each

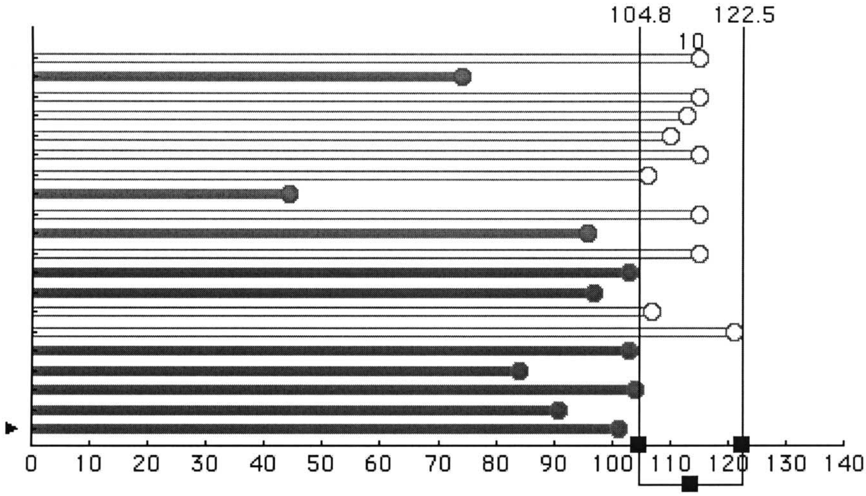

FIGURE 2 Battery data with range tool shown capturing the top 10 values.

data set that fell within a certain range. However, when Ceasara gave her explanation, it was clear that I had misinterpreted their activity. This could be attributable in part to my brief interaction with the students, but also in part to the fact that I had interpreted their activity in a manner that fit with how I would attempt to analyze the data. As a result, Ceasara's explanation was unanticipated. This was a pivotal event, placing me in the position of first ensuring that I understood the students' solution and then reacting to their explanation. I was therefore no longer able to proactively orchestrate the whole class discussion.

Although it could be argued that Ceasara and her partner did in fact partition the data with the range tool, the particular argument Ceasara offered did not fit with my interpretation of an argument based on partitioning. Ceasara stated that she used the range tool to identify the top 10 batteries out of the 20 that were tested and noted that *Always Ready* were more consistent with 7 out of the top 10 [12]. For me, this would involve attention to the proportion of each data set that fell within a certain range. However, the choice of the top 10 was, for Ceasara, a valid basis for selecting *Always Ready* as the better battery. I, however, was further confused by her use of the term *consistent* while simultaneously focusing on the top 10. At this point I was wondering if I understood her correctly. For that reason, I asked Jamie to explain her understanding of Ceasara's solution to me [14–16].

14. Jamie: I understand.
15. Kay: You understand? OK, Jamie, I'm not sure I do. So could you say it for me?

16. Jamie: She's understanding, I mean she's saying that out of 10 of the batteries that lasted the longest, 7 of them are green, and that's the most number, so the *Always Ready* batteries are better, because more of those batteries lasted longest.

Jamie's interpretation of Ceasara's explanation also did not fit with mine. I was now focused on Ceasara's use of the term *consistent*. For me, this implied a smaller range. This was not, however, the explanation that Jamie gave of Ceasara's solution, nor did it fit with the notion of consistency. Jamie's explanation centered on the number of batteries within the top 10. At this point in the discussion, I felt that I was faced with either imposing my interpretation of how to more effectively use the range tool or simply calling on another student without attending to Ceasara's solution method. Fortunately, Jason was able to pose a question that allowed me to explicate my concern. Jason noted that if Ceasara had chosen to use the range tool to capture the next "bunch of close ones" (e.g., the next 4 batteries) then there would have been 7 of each brand represented within the range tool [18]. At this point, I built off of Jason's question and attempted to get Ceasara to justify her choice of the "top 10" instead of, say, 14 [19]. However, when she responded that her selection of the top 10 data values was based on "trying to go with half," I was unable to think of a question to challenge her further [23].

As I deliberated about how to proceed, Blake raised his hand to say that he solved the task a different way. He had used the value bar to partition the data at 80 hr and had reasoned about the parts of each set that fell above 80 hr [26] as shown in Figure 3 (see also Figure 5 in the Appendix).

In doing so, he had judged *Tough Cell* to be the better brand because he would "rather have a consistent battery that I know that'll get me over 80 hr than one that just try to guess" [30]. I followed up by asking Blake why he chose 80 hr as a cut point [31]. He stated that his choice was based on the fact that "most of *Tough Cell* batteries were all over 80" [32]. I interpreted him as viewing 80 hr as a lower limit [33].

In attempting to tease out important differences between Ceasara's and Blake's explanation, I frame both explanation in terms of Toulmin's scheme for argumentation (see Figure 4).

Ceasara and her partner analyzed the data and reached the conclusion that the *Always Ready* batteries were better [12]. When asked to give a warrant, or to explain how they reached their conclusion, Ceasara stated that she looked at the top 10 of the 20 batteries [20]. Jason prompted me to ask Ceasara for a backing, or a justification for her warrant. The backing she offered was that she "was trying to go with the half" [22]. A similar exercise with Blake's explanation finds his conclusion different. He chose *Tough Cell* [30]. Blake's warrant or explanation for his conclusion was that "all of the *Tough Cell* is above 80" [30]. When asked to provide a backing or a justifi-

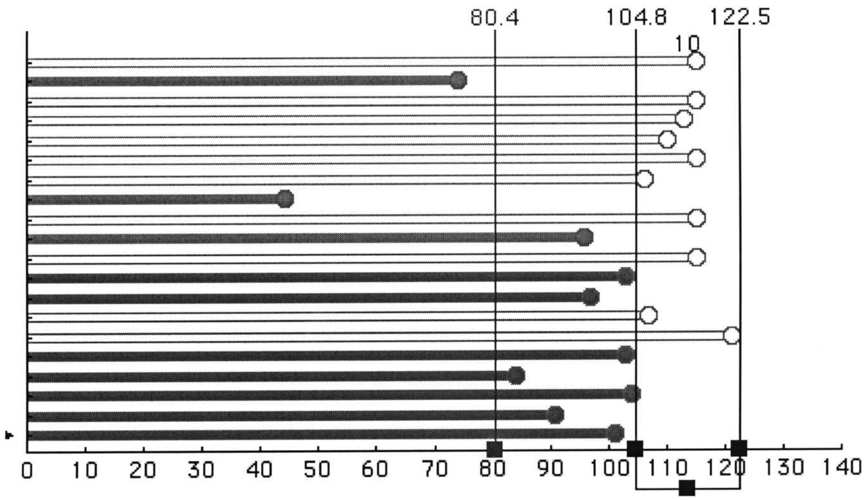

FIGURE 3 Battery data in second minitool with value bar and range tool shown.

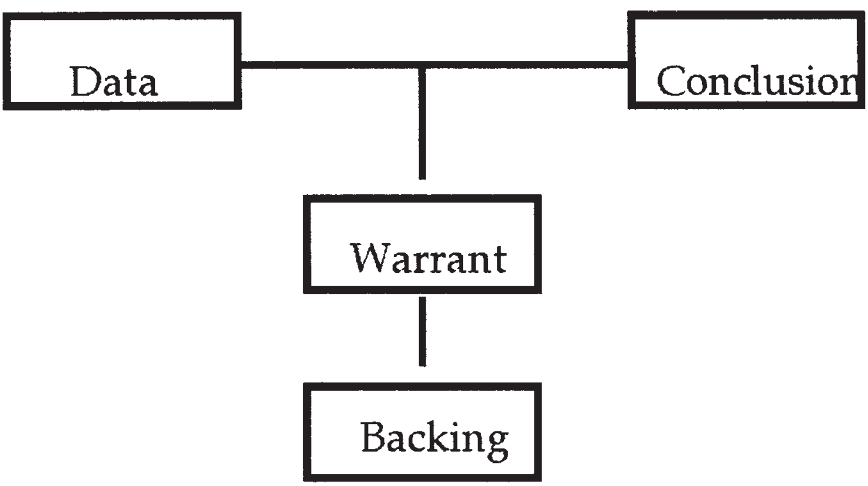

FIGURE 4 Toulmin's scheme for argumentation.

cation for his warrant, he explained "I'd rather have a consistent battery that I know that'll get me over 80 hr than one that just try to guess" [30].

A comparison of the two backings offered offers an opportunity to highlight an important shift that occurred in the course of the discussion. Although the students understood Ceasara's procedure (looked at the top half of the data and noted which brand had the most results there), her choice of the top 10 did not appear to be a valid statistic for everyone. In particular, Jason argued that if you modified the position of the range tool and chose the top 14 batteries instead of the top 10, you would have seven of each brand because the next 4 batteries are *Tough Cell* [18]. Choosing the top 10 was an arbitrary choice, not valid for the investigation at hand. Blake, on the other hand, gave a rationale for choosing 80 hr that appeared to be valid for all of the students. He wanted batteries that he could be assured would last a minimum of 80 hr [30]. In particular, his backing was grounded in the situation-specific imagery of the longevity of the batteries.

In this discussion, the students were beginning to negotiate what constituted a sufficient backing or acceptable mathematical argument in this classroom. Although I made no attempt to clarify the need for a backing that is grounded in the context of the investigation, this does emerge from Jason's challenge. Jason wanted to know what was significant about choosing 10: why not 14? I, however, was able to capitalize on Jason's comment by questioning Ceasara about her choice of 10 [18]. In addition, the challenge by Jason also prompted me to ask Blake to justify his decision to partition the data at 80 hr [31].

The exchange continued with Jamie questioning Blake's choice of *Tough Cell* as more consistent [34]. Jamie claimed that *Always Ready* was also consistent. In retrospect, it appears that Jamie and Blake were using different criteria for judging consistency. In doing so they were focusing on different features of the computer minitool. To clarify this difference, it is necessary to explain that on the previous task another student, Will, first introduced the term *consistent*. Students had been asked to judge the comparative safety of two makes of cars by looking at the braking distances of 10 of each make. Will had based his argument on consistency stating that it would be important to know, each time you applied the brakes, about how long it would take to stop. Therefore, the more consistent data set was the better make of car. He had used the range tool to identify the extreme data values in each data set to determine the range and then to make comparisons. In the batteries task, Ceasara and her partner had structured the data using the same feature on the computer minitool and had reasoned that because 7 out of 10 *Always Ready* were in the top 10, it was more consistent. Although Jamie's recasting of Ceasara's solution did not involve the use of the term *consistent* [16], it appears that in her conversation with Blake she was referring to the data structured as in Ceasara's solution (e.g., with the range tool as shown in Figure 2). Evidence of this can be found in her reference to "only 3 out of 10" [36], which is the number of *Always Ready* batteries falling outside of the captured range. However, Blake was arguing for *Tough Cell* based on the fact that all 10 lasted longer than 80 hr. Blake's response to Jamie was based on his way of structuring the data with the value bar, as he stated, "you still have 2 that are behind 80 in the *Always Ready*" [35]. It is only in retrospect that I became aware of my contribution to the lack of communication.

At this point in the discussion, there is every indication that Blake and Jamie are talking past each other—there is a breakdown in communication. Blake and Jamie have different understandings of the term *consistent* and are applying them to the data structured in two different ways. I contributed to the miscommunication by leaving both features of the minitool visible (e.g., the range tool and the value bar) as shown in Figure 2. This constrained the students' ability to attend to the two different ways of reasoning about the data. Blake's interpretation of consistency was based on the number of data values greater than 80 hr as defined by the value bar. Jamie's was based on the number of data values in the top 10 as captured by the range tool.

As the conversation continued, the lack of communication was further complicated by the fact that for Blake, the batteries being analyzed were a representative sample of the population [37]. I, in fact, recast his interpretation in terms of being representative [38]. However, Jamie, who was still focused on the top 10 batteries, did not understand why a similar thing would not happen with the *Tough Cell* batteries [39]. For her, the data sets, as inscribed on the computer minitool, were the population. This appears to also be the case for both Jessica and Sequoria, who spoke about getting "one of the bad batteries" [42] or "the ones that were lowest" [48].

Throughout this episode, there was evidence of the breakdown in communication between the students and myself. Initially, I was trying to make sense of the offered solutions for myself. This occurred after Ceasara's explanation [15] and again after Blake's [33]. In the latter case, I rephrased Blake's decision to partition the data at 80 hr in terms of a lower limit. In this instance I was verbalizing my understanding of Blake's method. However, I made no attempt to ensure that other students understood these explanations. This lack of attention to students' understanding is further highlighted by analyzing the exchange between Blake and Jamie that occurred at the end of Blake's explanation [34–40].

34. Jamie: Um, why wouldn't the *Always Ready* batteries be consistent?
35. Blake: Well, because all your *Tough Cell* is above 80, but you still have 2 that are behind 80 in the *Always Ready*.
36. Jamie: I know, but that's only 3 out of 10.
37. Blake: No, but see, they only did, what, 10 batteries? So the 2 or 3 will add up. They'll add up to more bad batteries and all that.
38. Kay: Oh, I see; as you get more and more batteries, it's going to get more, more bad ones if that's representative. OK, is that…Jamie?
39. Jamie: So why wouldn't that happen with the *Tough Cell* batteries?

40. Blake: Well, because the way that hose 10 batteries show on the chart that they're all over 80 that means that it seems to me that they would have a better quality.

Although I was aware of the sophisticated nature of reasoning about samples, I made no attempt to ensure that either Jamie or other members of the class understood Blake's explanation.

This lack of communication is consistent throughout the first part of the episode and is confounded by the tool use. I made no attempt to clarify which way of structuring the data the students were referencing in their arguments. This is a major contributing factor to the ineffective nature of the discussions. Because both structuring features remained visible throughout the discussion, there was a continued lack of clarity in the arguments. Students only saw the aspects of the inscription that were pertinent to their argument.

However, as the episode came to an end, a shift occurred in my focus on students' understanding as I attempted to ensure that they understood Jessica's comment [43]. I also made an implicit statement to the students about their responsibility to listen and attempt to understand others' offered solutions when I asked Sequoria to restate what she thought Jessica said [47]. This was motivated by my acknowledgment of the importance of Jessica's comment with respect to the agenda for the lesson. Jessica focused on features of the data sets that included attention to the extreme data values in each data set [42]. She was able to point to characteristics of the data that would be important in deciding the better brand of battery.

Initially, I had interpreted the students' contributions against my more sophisticated understandings of the tool and the task. I was giving too much agency to their activity on the computer tool. As an example, when Ceasara ordered the value bars, I saw the shape of the distribution and assumed it was transparent for her also. In coming to understand the importance of taking account of their actions and constructions, I began to work much harder at making sense of their activity with the tool instead of imposing my understandings.

Transition to the *AIDS* Episode

In the weeks following the *Batteries* episode, as the students engaged in a sequence of data analysis tasks, I continually reflected on my activity against the background of the students' learning and the overall intent of the instructional sequence. This occurred through daily journal writings and interactions with other members of the research team. In doing so, I came to understand the importance of deliberately facilitated whole class discussions and the norms for argumentation that would support students' active participation in these discussions. A critical feature of these aspects of the classroom was the role of the computer minitools in supporting my efforts. Their design had been a result of efforts to provide means of supporting students' ways of reasoning statistically. For that reason, they offered a necessary resource for achieving the mathematical agenda. Without them, my ability to support shifts in the students' ways of reasoning was limited. It was therefore necessary that I build on the affordances offered by the tools. As a result of my continually developing understandings, there was a gradual shift in my ability to get beyond my interpretations of the task and focus on the interpretations of the students and their use of the tools. In doing so, I acknowledged the importance of communication between the students and myself and the role of the tools in supporting that communication. This shift in my understanding of the importance of the students' ways of reasoning can be seen in the *AIDS* episode that occurred 23 class sessions later. In this episode, I was able to build from the students' analyses to advance the mathematical agenda.

AIDS Episode

The task for the students in the *AIDS* episode varied from the one in the *Batteries* episode. Over the course of the teaching experiment, the research team decided that asking the students to choose the better brand or better treatment focused the students' analyses on the choice. By asking the students to think of a way to structure and organize the data so that someone else could make a reasoned decision, the students were asked to investigate patterns in the data and then decide how best to represent those to someone who would not see the data sets in their entirety. In this way, discussions focused on features of the students' inscriptions. However, the inscriptions that the students developed were modifications of the data as structured by using features on the second minitool. As an example, one option the students used was the create-your-own-groups method to determine the actual number of data values above and below a cut point. The resulting inscription would be a line with the location of the cut point noted, the extreme data values marked, and the number of data values appearing in each interval recorded, as shown in Figure 5.

When making comparisons, students would stack these (similar to the way they were stacked on the computer minitool) and reason across them to judge which brand or treatment was better. This judgment would be based on how the data were distributed to the two intervals. In most situations, it was not unusual for all of the students to agree on a brand or a treatment and still discuss at length the best way to represent the results of their analyses. The discussion therefore focused on features of the students' inscriptions. This shift towards what we came to call a data-analyst's perspective proved significant in supporting students' ongoing data investigations.

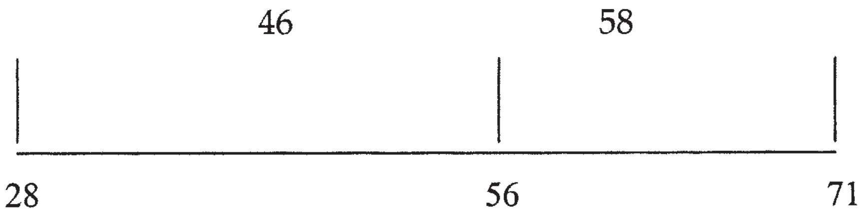

FIGURE 5 An inscription resulting from using the create-your-own-groups option.

During the introduction of the AIDS task situation, it became evident that the students were quite knowledgeable about AIDS and understood the importance of finding an effective treatment. Furthermore, they clarified the relation between T-cell counts and patients' overall health (increased T-cell counts are desirable). They had studied AIDS in health class and were aware of social issues surrounding treatments and the importance of a cure. After the discussion, I introduced the data inscribed in the second computer minitool.

At this point in the instructional sequence, the research team was interested in supporting shifts in students' ways of reasoning toward structuring the data multiplicatively. This particular task was selected because the two data sets contained an unequal number of data points. In particular, one data set was much larger than the other (46 patients compared with 186 patients). In earlier tasks with unequal numbers of data points, students spent a great deal of their time deciding which data points to ignore to have equal sample numbers. One goal of this task was to create a situation where equalizing the sets by eliminating points was not possible. We hoped that this would create the need for multiplicative ways to structure the data to make a comparison, because direct additive comparisons would be inadequate.

As the students worked in pairs on their analyses, I again monitored their activity to plan for the subsequent whole class discussion. In this particular instance, the discussion of the data creation process and the students' analyses encompassed the entire class period. As a result, I was able to collect the students' reports and inscriptions and use them in making decisions for the following day. In reviewing the students' solutions, we noticed that some students found qualitative ways to make comparisons between the two data sets, such as referring to where the "majority" of the data were located. Other students found quantitative ways to make the distinctions by noting the number of data values above and below a certain cut point. Still others had structured the data into four equal groups and reasoned about the percentages of each data set that fell above a certain T-cell value. We therefore decided to begin with the qualitative inscriptions and move toward the quantitative ones in hopes of shifting the students' ways of reasoning toward inscriptions that made multiplicative ways of reasoning explicit. In particular, we anticipated being able to problematize the reports that noted the number of data values above a T-cell count of 525 or 550, because direct additive comparisons were insufficient (see Figure 6).

As an example, just giving the count of data values above 525 (e.g., reasoning additively) would lead one to conclude that the traditional treatment protocol was more effective. Only when that number is taken as a proportion or a percentage of the total can an adequate argument be made (e.g., reasoning multiplicatively).

In making these a priori decisions about how to sequence the students' reports, we made a conscious decision to start with qualitative comparisons and then move to quantitative ones. Our purpose in this structuring was to start with analyses based on perceptual patterns in the data that provided a means for all students to engage in the critique. By building from the students' inscriptions, we hoped to initiate shifts toward more sophisticated inscriptions, thereby providing an opportunity for all students to participate. This is in contrast to introducing a four-equal-groups inscription and spending the class period explaining it to those who do not understand. We judged that the former alternative provided us with the means of supporting the students' developing understandings.

I began the whole class discussion the next day by asking the students first to decide if they could understand each report and then decide if they thought the inscription was an adequate way to represent the data to enable someone who had not seen the data sets to make a decision [50]. In doing so, I was attempting to shift the conversations from simply providing explanations of methods to critiques. To facilitate this process, I reproduced the students' inscriptions on chart paper. The creation of these artifacts allowed the class to see the reports clearly and removed ownership during the discussion. Although the authoring students might have been able to determine which inscription was theirs, other students in the class could have solved the task in the same way. For this reason, the students did not attend to whose solution was being critiqued.

The discussion began with Jamie giving her assessment of the first report [56]. She determined that the report was adequate because she could tell "where the range[4] is starting and ending" and "where the majority of the numbers are" [52]. This justification created a problem for Derrick, who did not understand what Jamie meant by the term *majority* [57]. In the ensuing discussion, I authorized the turn taking as the students attempted to answer Derrick's question [59–74].

59. Jamie: Where most of the numbers were.
60. Derrick: Where most of the numbers are…
61. Kay: Sheena, can you help?

[4]In this turn, Jamie is using the term *range* to refer to the range of values. Students use of the term was very casual and often conflicted with the statistical definition.

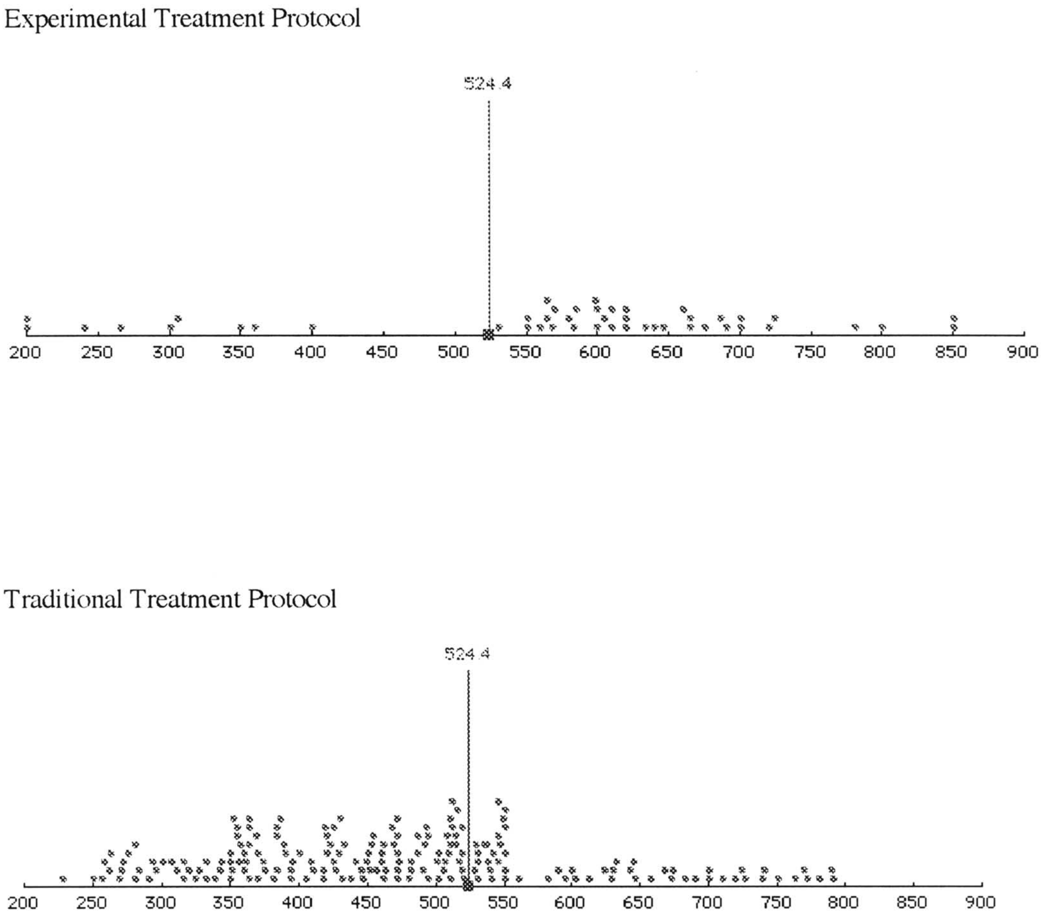

FIGURE 6 AIDS data in second minitool partitioned at 525.

62. Sheena: Like, when she talks about like when she says, like when you say where the majority of the numbers were, where the, where the point is, like, you see where it goes up?
63. Kay: I do see where it goes up.
64. Sheena: Yeah, like right in there, that's where the majority of it is.
65. Kay: OK.
66. Derrick: The highest range of the numbers?
67. Sheena: Yeah.
68. Kay: The highest range?
69. Sheena: Oh, no.
70. Vallory: No.
71. Kay: Vallory?
72. Vallory: However many people were tested, that's where most of those people fitted in, in between that range. With those little…
73. Paul: You mean this range here?
74. Sheena, Jamie, and Vallory: Yes.

From the comments, it appeared that for some of the students, the inscription was a representation of the distribution of the data. In other words, they could determine where the data were with respect to the inscription. In particular, Jamie referred to "where most of those people fitted in" [72]. This was important because our goal was that the students build initial understandings of distributions from perceptual patterns in the data.

After this discussion, I introduced the next two reports as shown in Figure 7 (see also Figure 10 in the Appendix).

Both of the groups generating these reports initially had structured the data in a similar manner by using the computer minitool. One had partitioned the data at 525 and the other at 550. Both had then reasoned about the number of data values on either side of the cut point, albeit one in qualitative terms (e.g., majority) and the other in quantitative terms [55]. At this point, Vallory asked why the one group had chosen to partition at 550 [78]. She noted that 550 did not represent the median, so she could not determine its importance. Here Vallory was asking for a backing for the warrant. Again, using Toulmin's scheme, we note that the students had analyzed the data to reach a conclusion. Their warrant consisted of looking at the number of data values above and below a T-cell count of 550. What was not obvious from the report was the choice of 550. In other words, the report had not included a backing for the warrant. Vallory wanted to understand why the group had decided to partition the data at a T-cell count of 550 because it did not represent any particular value of the data set such as the median. The norm that had become instantiated in the classroom was that of justifying in terms of the context or the way the data were distributed. Vallory was making it clear that she could not ascertain the justification from the report.

Report #2:

The new drug was better than the old. The majority of the old ones are behind 550 and the majority of the new drug was in front of 550.

Report #3:

Experimental treatment	Traditional treatment
225 - 525	**225 - 525**
9	**130**
525 - 850	**525 - 850**
37	**56**

FIGURE 7 AIDS data as noted in Reports 2 and 3.

At this point, one of the students who had solved the task in a similar manner noted that, when working on the computer minitool, she noticed that the data "lined up straight on 550" [88]. I recast this as basing the cut point on a natural break in the data [89]. This was an important concept. One of the goals of the teaching experiment was that students would view distributions in terms of shape. Here Megan was offering an explanation based on the shape or breaks in the data as inscribed on the minitool. For that reason I wanted to highlight both the solution and Megan's explanation. Paul further clarified this choice by stating that these students had chosen the T-cell count of 550 because the hill of one data set was mostly below this value and the hill of the other was mostly above [90]. This is a significant exchange because it points to a shift in the norms for argumentation. The students would not accept an argument whose backing could not be justified in terms of the analysis of the data.

It is important to note that Vallory asked for the backing for the choice of 550 and not 525. In looking at the inscription with a cut point at 525, it could be argued that the data were partitioned on the basis of the midpoint of the extremes (e.g., finding the average of the maximum and minimum value). In previous classes, students had attempted to use the midpoint of the two extreme data values as a statistic for describing data sets. They called this value the "middle of the range." I had worked to problematize this way of reasoning, because it has no statistical value in terms of distributions. In this episode, however, the students accepted it as a valid cut point because it coincided with the break in the hills. We were aware of how the students had reasoned in creating the inscription but appropriated the inscription to allow the opportunity to compare and contrast the qualitative and quantitative inscriptions shown in Figure 7 [91].

It in fact proved to be a pivotal move to juxtapose the second and third reports (see Figure 10 in the Appendix). Although I had recast the two inscriptions as partitioning the data around the break in the hills [91], one group described the results in terms of where the majority of the data fell and the other used the computer minitool to determine the number of data values that fell above and below the cut point. Juxtaposing these two reports made it possible to problematize the use of direct additive comparisons. This was facilitated by Marissa noting that the second of these two reports would be more confusing, "since the old program has more numbers than the new program" [92]. I realized the importance of this contribution and recast it so that all members of the class could focus on the possible problem [93–95].

93. Kay: Oh. So it looks like that there's more. They had 56 that were above 525, and they only had 37?
94. Marissa: So it's like, I guess what I'm trying to say is it's harder to compare them.
95. Kay: What about what Marissa said? She just said there were more people in the old program so if you actually looked at the actual numbers of people, you find out that they had 56 that were in this upper range, which is where we want to be, and these only had 37. So somebody might say the old program was better because there were more. Jamie?

As the discussion continued, students were able to acknowledge the problematic nature of the report but could not determine a solution. Finally, at Kyle's direction, I created a modified graph of the data as shown in Figure 8 (see also Figure11 in the Appendix).

Paul then restated the initial quandary by referring to the graph [99]. At this point, students offered a variety of ways to reason about the numbers [101, 103, 112, and 116].

101. Blake: But then there's more people with the old program than there is with the new program.
102. Kay: Jason.
103. Jason: Then you see that there's 37 is more than half over 525 and 56 is not more than half of 130…more of them on the bottom than on the top.

FIGURE 8 Graph created from AIDS data.

112. Jason: OK, you see how 37 is more than half of 9 and 37 together? But 56 is not more than half of 30, 130, and 56 put together. There's more on the bottom one than on the top one.
113. Kay: OK, who can help me out with that, who can say that a different way so that I might could understand that? Will, can you say it a different way?
114. Will: Well, in that situation it wouldn't matter how many people were in there because see like…
115. Kay: Big voice, Will.
116. Will: On the bottom one you have, see what Jason was saying there's more than there is below 525 and so that means that that one is better because the top one it doesn't even have close to half of what the one below 525 is on that one. So that means that if, if that was the same amount of people it had like, if they both had the same amount of people and, but, and they had the numbers and everything, and this one, the bottom one was a however much more that of…

Blake had an intuitive understanding of the problem but was unable to clarify other than to point to the difference in the number of patients in each program [101]. Jason reasoned that because more than half of the patients in the experimental treatment protocol had T-cell counts greater than 525 and less than half of the patients in the traditional treatment protocol had T-cell counts that high, the experimental was better [112]. The issue was reconciled when Paul suggested that the students use percentages. Their initial conjectures were confirmed when they made these calculations.

The last report to be shared in the whole class discussion involved students structuring the data into four equal groups, as shown in Figure 9.

After I posted the report on the white board, Blake stated that he felt it was adequate because the extreme data values were similar in both data sets [127] and "with the four equal groups, you can tell where the differences in the four groups" [129]. In his explanation, Blake pointed to two mathematically significant issues, both of which were of interest to me. The first concerned how to interpret the four-equal-groups graph. Blake noted that the new treatment was better because "the three lines for the equal groups" [135] from the experimental were above 525 compared with "only one of them" [137] for the traditional treatment. Although Blake's explanation was somewhat difficult to understand, I interpreted it to mean that he noted that 75% of the experimental data (i.e., three lines) were above 525, whereas only 25% of the traditional data (i.e., one line) were above 525. I believed that many of the students in the class did not understand Blake's offered explanation. Therefore, teasing out the meaning of the graph was an important aspect of the continuing discussion.

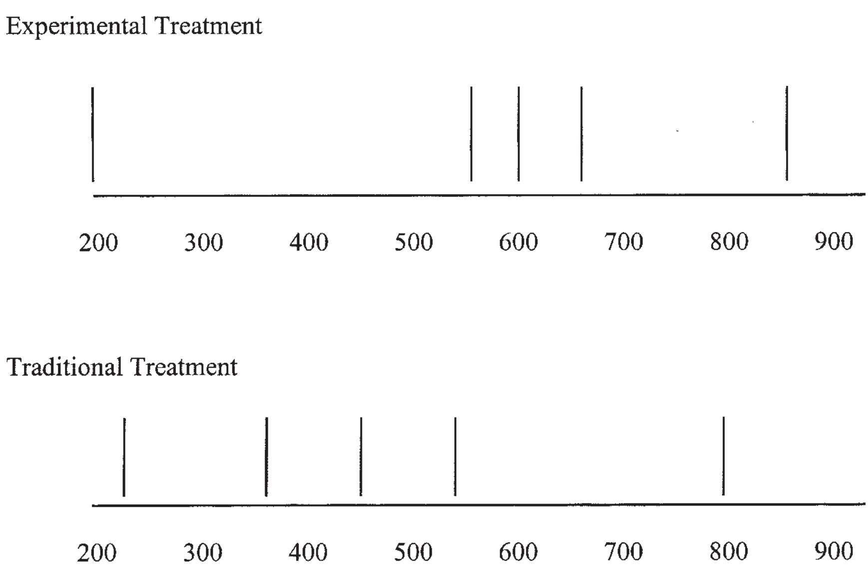

FIGURE 9 Inscription of the AIDS data organized into four equal groups.

The second mathematically significant issue concerned Blake's reference to the fact that because the extreme data values (maximum and minimum) were nearly the same on the two data sets, the comparison was easier [127]. This was echoed by Marcus, who noted that in the past they had not been similar (i.e., "like kind of crooked" and "this time it's easier to see because it's right under each other" [139]). The issue of similar extreme data values had occurred on prior tasks, and in some cases the students had stacked the two inscriptions and aligned the minimum data values in both data sets, even when they were not the same. This was clearly problematic from my perspective, because the values were on a scale that could not simply be adjusted so that these two values were always the same. Here the students noted that this inscription was easier to read because of the similarity in extreme data values of the two data sets, in particular the minimum value. Although it would have been difficult for me in this class session to pose a comparison problem where the minimum data values were very different to make that an explicit topic of conversation, I nonetheless noted it as an issue that needed to be addressed.

Clarifying the student's understanding of the four-equal-groups graph was therefore the primary consideration. As I considered how to continue, Marcus suggested placing numbers in the intervals to indicate the number of data points in each [139]. I found this very significant, because the purpose of the four-equal-groups graph was to remove the numbers and thus the problems of comparing data sets of unequal size. However, it was obvious from the comment that the intervals did not represent proportions of the data for him. I therefore judged that at this point, other students were unable to read the four-equal-groups inscription. Not only did it not represent the data, they had no resources for recreating the data.

As the lesson continued,[5] I therefore asked the students to calculate the actual number of data points that would fall within each interval in each data set. As they did, I wrote each of these values within the intervals of the four-equal-groups inscription as shown in Figure 10 (see also Figure 12 in the Appendix).

Once these numbers were visible, the students commented that the numbers did not help because there were not the same number of people in each treatment. This was similar to the argument that Marissa made about the third inscription [92 and 94]. I then asked the students if they knew what percentage of the data were in each interval. In their answer, the students were able to clarify that they understood that 25% of the data set fell within each of the four intervals. It could be argued that this was facilitated by the fact that they had just used calculators to find 25% of each data set to place the numbers in the intervals. Nonetheless, I then noted on the inscription, at the students' direction, that 75% of the patients in the new treatment program were in the same range of T-cell counts as only 25% of the patients in the old treatment (see Figure 11 and Figure 12 in the Appendix).

[5]This part of the lesson is not included in the information in the Appendix. However, to clarify my role in this lesson, it is necessary that I extend my analysis to include an additional portion of the lesson.

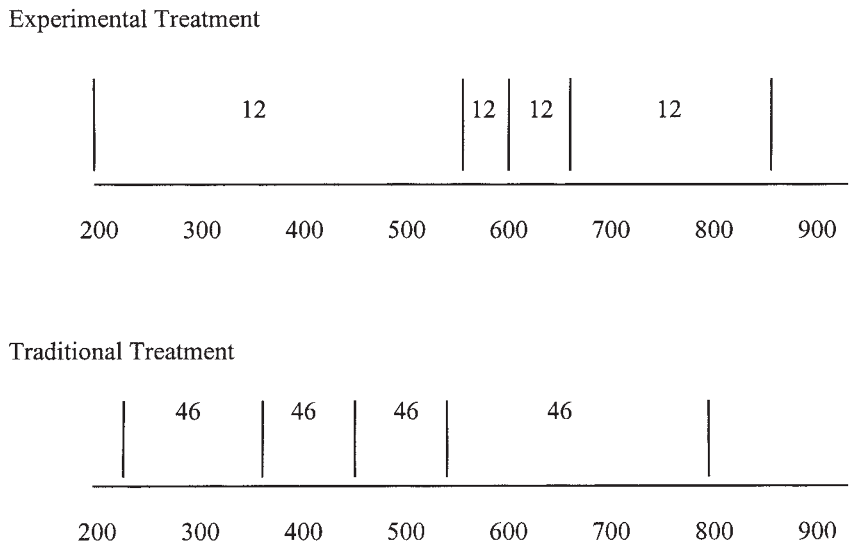

FIGURE 10 Four-equal-groups inscription with number of data values noted.

For many of the students, the process of calculating the numbers and then going back to the percentages was necessary to build an understanding to underlie the inscription. They needed to know where the data were behind the inscription. The four-equal-groups inscription was not interpreted as a providing information on the distribution of the data in the same way the first inscription was. It was therefore important for them to understand how the data were being represented.

Perhaps the most critical aspect of my role and that of my colleagues in this particular episode was the a priori selection and sequencing of the students' inscriptions of the data to be discussed. During this whole class discussion, a planful attempt was made to sequence the solutions to provide a progression toward more sophisticated ways of structuring the data. A review of the sequence of solutions finds that the whole class discussion initially focused on reports that structured the data by partitioning around the hills. The first two solutions offered a qualitative distinction between the two data sets by focusing on the location of the majority of the people. The third, however, gave a quantitative distinction that provided the opportunity for a problem to arise from direct additive comparisons. This, in turn, offered me the opportunity to problematize the direct additive comparisons and shift the discussion to more sophisticated ways to reason about the data sets. The four-equal-groups solution was then a logical next step in the progression. In this way, the inscriptions served as tools for reorganizing the students' activity, providing me with the means of supporting their mathematical development (Dörfler, 1993; Kaput, 1994; Meira, 1998; Pea, 1993).

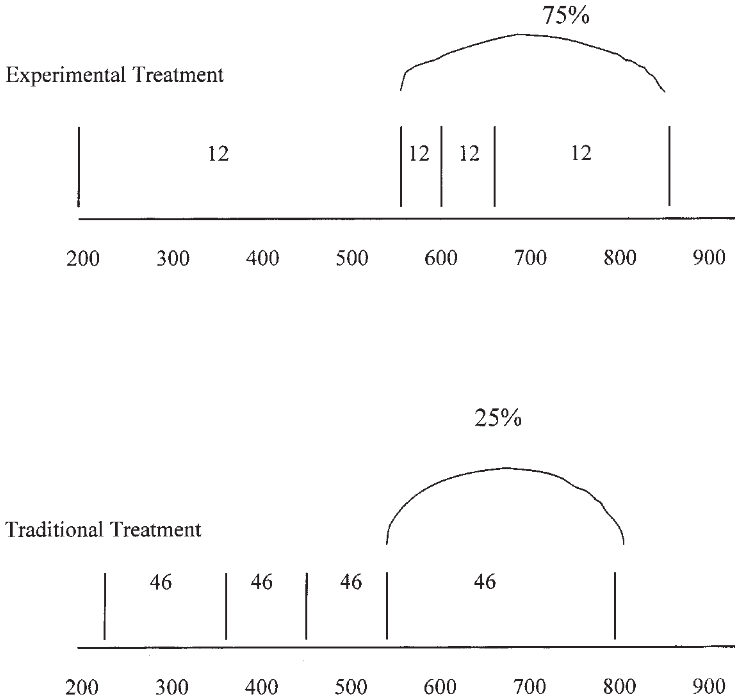

FIGURE 11 Four-equal-groups inscription with percentages marked.

Throughout this process, there is also evidence that I am cognizant of the importance of ensuring that students understand the discussion. This is supported by the students' understanding of their obligation to ask questions if they do not understand. This is a shift from the earlier episode, where I focused on my understanding. The importance of this aspect of the teacher's role is highlighted in this episode by the productive nature of the class discussions when ensuring that everyone understands is an accepted norm.

DISCUSSION

The importance of the role of tools and inscriptions as a means of supporting my efforts to achieve the mathematical agenda is grounded in initial decisions made by the research team before entering the classroom. Those decisions were intended to provide resources for supporting students' emerging understandings while taking their initial understandings as starting points for instruction. For that reason, the research team was guided by the premise that the computer tools would be critical in supporting our mathematical agenda. Students would need flexible, efficient ways to structure and organize sets of data as they conducted their investigations. Our initial dilemma was centered around a concern for offering means of support while not providing so much support that the students' activity became that of following anticipated cues.[6] As a result, the approach we took when designing the computer-based tools offers particular ways of structuring data that are designed to fit with students' current ways of understanding, while simultaneously building toward conventional graphs (Gravemeijer, Cobb, Bowers, & Whitemack, 2000). This was intended to facilitate my ability to build from students' contributions in the course of discussing the students' analyses.

The two computer tools and the instructional tasks therefore were designed to provide resources for the teacher to use while interacting with the students. An underlying premise was that, during deliberately facilitated whole class discussions of students were analyses, these tools would provide the teacher with means of supporting the students' mathematical development. It is also important to note that the research team made assumptions about the classroom participation structure that would support such development. For this reason, I was balancing attention to norms, the use of tasks, and investigations with the tools in the course of my interactions with the students. In doing so, I was working to stay focused on the mathematical endpoints as defined by the conjectured learning trajectory.

My overarching mathematical goal was to support students' ways of reasoning about data in terms of distributions structured multiplicatively. Although I viewed this as an endpoint to be aimed for, the detailed route to the endpoint was not predetermined. In planning the instructional sequence, the research team had developed a hypothetical learning trajectory that guided the development of instructional materials (e.g., the tasks and the computer-based tools). However, this trajectory was continually subject to modifications based on ongoing analysis of the students' daily activity. As a result, revisions were made against the background of the hypothetical trajectory but with the information provided by informal analyses of students' activity with the tools. Similarly, my decision-making process in the classroom was informed by the mathematical agenda but constantly was revised and modified in action based on the students' contributions. These contributions were grounded in the students' use of the tools and the resultant inscriptions that they produced.

Part of my decision-making process was informed by my anticipations of the students' ways of reasoning on each task. Initially, I had trouble distancing myself from my own ways of reasoning and taking account of the students' current understandings. For that reason, I often imposed more sophisticated meanings on their use of the tool, thereby creating problems in my interpretations of their analyses. I had the advantage of seeing the endpoint of the trajectory and being able to envision how I might traverse the route via investigations on the tools. What I did not

[6] A discussion of this quandary can be found in the Appendix and is grounded in Doerr's (1995) distinction between expressive and exploratory computer environments.

account for were the diverse ways of reasoning that the students used in their initial investigations. Nor did I acknowledge that for many of these students, the activity was not about analyzing data—it was about completing tasks in school. I was, in fact, giving agency to the tools instead of acknowledging the importance of accounting for the students' activity as they used the tools for analysis. In addition, the norms as they were initially constituted in the classroom did not support my efforts. Only through a process of constant renegotiation did it become possible to facilitate the type of discussions that I initially had envisioned. Only after I shifted my focus to the students' activity as supported by the emerging norms did the tasks, tools, and inscription function as means of supporting my agenda.

Both Gravemeijer (1998) and Simon (1997) have talked about the cycles that occur in the development of instructional materials and in teaching, respectively. Gravemeijer discussed the iterative cycles of revision in designing materials that lead toward a local instruction theory. Simon discussed how the cycle of teacher's knowledge, a conjectured learning trajectory, and interaction with students eventually support mathematical growth. The episodes described here offer a detailed account of this process by highlighting the ongoing decision-making process of one teacher as I interacted with my students. These decisions were informed by my knowledge of both the students and the mathematics and were framed against the hypothetical learning trajectory. An important part of this trajectory was grounded in the design and development of tools for supporting its emergence.

The way of conceptualizing the teacher's role that I have described in this article is not intended to make light of the extreme complexity of teaching or to be seen as a critique of teaching. My intent, instead, is to acknowledge the complexity while attempting to provide a frame for thinking about the numerous and diverse aspects of teaching and tools for supporting those goals. In Simon's (1997) description of the mathematics teaching cycle, he noted the "inherent tension between responding to the students' mathematics and creating purposeful pedagogy based on the teacher' s goals for student learning" (p. 76). In a similar manner, I am attempting to point to issues of design and pedagogy that inform the decision-making process while acknowledging that the mathematics must be the central focus in all decisions.

ACKNOWLEDGMENTS

The research reported in this article was supported by the National Science Foundation under grant REC9814898 and by the Office of Educational Research and Improvement through the National Center under grant R305A60007. The opinions do not necessarily reflect the views of either the foundation or the center.

I wish to thank Anna Sfard, Ellice Forman, Shelly Goldman, and Phil Bell for their helpful comments on earlier drafts of this article.

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