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Representational Tools and Mathematical Understanding

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Understanding how cultural mediators and social interaction promote meaningful learning requires that each student's perspective, reasoning, and construction processes be taken into account. In my analysis of the classroom episodes, I consider individual students' progress as they use tools, discuss data distributions, and interact with their teachers and their peers. I argue that data display tools provide a partial context for discussions but do not constrain the students' interpretations or the way they reason about the data. Students' approaches to the mathematical relations discussed in the classroom result rather from the meaning they attribute to the different features of the displays, the teachers' questions, and the evolving interaction with their peers.

In mathematics classrooms, students encounter a wealth of tools, procedures, and culturally developed representational systems. Tools, artifacts, and cultural representations are important components of mathematical learning. But they are only part of the story.

Vygotsky and Luria (1993) considered that tools, artifacts, and cultural representations may introduce new mental functions connected to their use, but they also may abolish natural processes, replacing and reorganizing certain "simplest psychological functions" (Vygotsky & Luria, p. 169). It is likely that some tools such as computer software, mathematical notation, formulas, and rules to deal with mathematical problems may sometimes hinder higher order functions and logical reasoning processes by making them unnecessary. Consider, for instance, children's learning of arithmetic algorithms or of the rules for solving algebraic equations. These powerful mathematical tools may allow students to solve equations and computation problems easily and rapidly and, if explored in depth, certainly can contribute to the emergence of sound understanding of mathematical princi-

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ples. However, unless students are guided to explore these principles, they risk learning procedural rules without the proper mathematical understanding implicit in its procedural steps (see, e.g., data by Nunes, Schliemann, & Carraher, 1993, and by Schliemann, 2000).

Tools do not directly determine ways of reasoning. Mathematical structures embodied by the tools and symbolic systems do not transfer directly to the user's mind and, as pointed out by Vygotsky (1978), "Cognitive activity is not limited to the use of tools or signs" (p. 55). As Cole and Griffin (1980) rightly suggested, cultural tools may lead to larger productivity and more efficient and organized use of mental functions but do not amplify one's mental capacity. As we argued elsewhere (Carraher & Schliemann, 2002), in the analysis of an educational software tool,

> We view the software as merely providing a context for discussing mathematical concepts. Although the software renders quantities as crisp visual images on the monitor, there is no reason to expect that the students will understand the concepts merely by inspecting the diagrams and notation. On the contrary, every configuration of objects on the computer screen was open to multiple interpretations. One of our main points is that, at almost every turn, the students are trying to reconcile what they observe happening on screen with rich interpretive schemes. (p. 6)

Given the complex interaction between the use of tools and the development of reasoning and learning, the question that should concern educators is not how powerful or effective cultural tools are in promoting learning, but rather what teaching practices and classroom interactions can promote meaningful learning and understanding of the mathematical principles and relations embedded in cultural tools and representations. To paraphrase Papert's (1990) comments on the role of computers in education, the issue is not to determine the effect of tools in learning and education but to remake and rethink classroom activities in the presence of new cultural tools.

Learning involves an interaction between individuals and their sociocultural milieu. As others have noted (e.g., Cobb, 1995; Cole, 1996; Hatano & Inagaki, 1992; Resnick, 1994; Saxe, 1991), constructivist and sociocultural accounts of learning and development are not alternative but rather complementary views. Many would agree that individual construction and sociocultural contributions are equally important, deeply intertwined aspects of learning. However, as researchers shifted their focus from individual knowledge construction to the sociocultural character of learning and development, they sometimes lost sight of the role of an individual's logical reasoning processes and of their previous experiences and understandings.

Attempts to understand the role of cultural tools and social interaction in learning and development benefited greatly from the work of Luria, Vygotsky, and others in the Russian cultural–historical tradition. I believe, however, that the analysis of children's learning as they deal with cultural tools in interactive settings could be refined by Piaget's views on how social interaction and cooperation can promote learning and development. I therefore adopt a Piagetian perspective in this article to analyze individual and sociocultural aspects of our target episodes. It may seem surprising to choose Piaget's theory for this task. Piaget's analyses are based on individual interviews with children, and, apparently, his theory does not assign a major role to sociocultural factors in human development. Nonetheless, Piaget (1965/1995) provided a wealth of empirical findings and theoretical ideas concerning peer interaction and cooperation that are relevant to the analysis of interactive classroom events.

As is widely known, Piaget believed that mathematical understanding comes not from a passive perception of the physical properties of the objects but from children's reflection on the actions they perform on the objects (Piaget, 1973). Mathematical understanding is the result of a construction by the individual, not a mere internalization of what is offered by representational tools or by teachers. However, as he further elaborated (Piaget, 1965/1995), this construction takes place in social settings, where the cooperation between individuals can become the object of reflection by others. For effective interaction and cooperation to take place, participants must use similar systems of signs and meanings and carry out similar operations. Neither the individual nor the set of individuals modifies individual mental processes. What matters in the development of logical reasoning are the continuously modified relationships among individuals and the type of social interactions thus created. Whereas social interactions based on constraint typically lead to superficial notions, cooperation, as a system of interpersonal actions governed by the laws of equilibrium and organized into reversible systems, allows for the development of objective and coherent systems of operations (Piaget, 1932, 1965/1995). In this process, each student's previous experiences, conceptions, and access to cultural tools contribute to shape his or her attempts to understand each new task and the others' points of view.

I see in the set of video clips a sequence of activities aimed at promoting students' understanding of relative frequencies and proportionality. In my analysis of the classroom episodes, I attempt to consider individual students' progress toward expressing understanding of these core mathematical concepts as they used the tools and as they interacted with their teachers and their peers. Drawing mainly on Piaget's analysis of social interactions and of cooperation in the emergence of logical and scientific reasoning (Piaget, 1965/1995; see also Schliemann, 2000), I attempt to consider students' individual views and reasoning processes while they interacted with teachers and peers and with different types of mathematical tools and representations. Where possible, I try to achieve the following:

- Infer, from students' verbal statements, their individual ways of understanding the mathematical relations embedded in data distributions produced by software tools or by the students themselves.

- Analyze how students' views and interpretations were interrelated and how the teachers' questions and proposals led to or prevented the emergence of new mathematical understandings.
- Identify how students' experience with the software tools they worked with did or did not contribute to the development of new mathematical understandings.

My analysis of each clip includes a brief overall description of the episode or subepisode I refer to. Then I highlight the events supporting my main points. At the end of each episode's analysis, I summarize these points in relation to individual understandings, peer–teacher interaction, and experience with software tools.

THE *BATTERIES* EPISODE

This first episode shows the students' early use of a software tool for representing data. The tool allowed them to represent previously collected data on the longevity of two brands of batteries. The teacher–researcher team designed the lessons to engender discussions where the students would ground their decisions on the visual representations provided by the tool. The teacher–researcher in charge of the classes (Kay) set the stage for the type of discussions she expected from the students, namely, discussions of how the data visual display could support decisions that would have everyday consequences: She asked the students to compare the results for the two brands and to recommend one of them for purchasing.

The first contribution of the software tool in this early episode consisted of organizing and displaying the data to be discussed. Because we do not know how students used the tool to produce the graph and what specific graphs each group produced, no further inferences can be made on how the artifact helped the students as they collected the data. My analysis therefore is limited to what happened when a representation produced by the tool was discussed with the whole class. I divide my analysis of this first episode into two main themes: the criteria to choose the better battery brand and the meaning of the measures displayed by the software tool.

Choosing the Better Battery Brand

When asked to choose the better battery brand and to justify their choices, the students focused on different characteristics of the software display and of the data distribution and hence adopted different views.

Caesara tried to explain why she thought that the *Always Ready* battery was better but found it difficult to express her view [8–15]. Jamie rephrased Caesara's point [16], apparently interpreting the software display in a similar way. Jamie's clarification allowed Kay to further focus on Caesara's views and to pose a new question [19] that forced Caesara to further explain her choices [20–22].

Caesara's strategy, as clarified by Jamie, seemed to consist of using one of the software features, selecting the top 10 data values, and then deciding that the better battery brand was the one more frequently represented in this subgroup. Her approach is analogous to how one would judge the performance of two teams of runners. The team with more runners in the top places wins the competition. From a statistical perspective, Ceasara considered only the batteries performing above the median. Caesara's choice was based on what could be considered as mathematical aesthetics: "half of 20 is 10" [20] and "I was trying to go with the half" [22]. She seemed to interpret the assignment as one that required "doing something with numbers."

Jason noted that, if the cutoff point were lowered, a different picture would emerge, with 7 *Tough Cell* batteries in the high-performance group [18]. He suggested that the choice of a cut was arbitrary and, as such, might lead to conflicting conclusions. A few minutes later Blake followed Jason's lead and asked the teacher to set the cutoff below (that is, to the left of) the bars representing the *Tough Cell* batteries [24]. He then remarked, "There's still green ones [*Always Ready*] behind 80, but all the *Tough Cell* is above 80. So I'd rather have a consistent battery that'll get me over 80 hr, than one that just try to guess" [30]. Jessica and Sequoria supported his view by pointing out that *Always Ready* batteries occasionally failed to meet the standard set by Jason [42, 48]. In statistical terms, these students were comparing the range of performance of the two battery brands rather than their central tendency.

Concerning students' individual understandings, the two types of approaches exemplify a tension between different views of mathematics. The first, adopted by Caesara and Jamie, is typical of the mathematics practiced in traditional classrooms where priority is given to the correct manipulation of numbers, regardless of the context numbers and computations are applied to. The second, favored by Jason, Blake, Jessica, and Sequoria, fulfills the teacher's request to consider the everyday consequences of their choices and reminds us of how people use mathematics to solve problems at work, where the contextual goals and meaning of the situations prevail.

As the discussion unfolded, Kay's questions and participation allowed students to take turns. Despite the different perspectives at play, the students collaborated to a certain degree, providing and explaining their views and questioning those of others. The discussion, however, seemed to largely follow two separate paths, and there was no evidence suggesting that cooperation occurred across the two different perspectives.

The software display provided common referents for the discussion and allowed students to clearly indicate the properties of the data they were attending to. The visual representation also guided students to intuitively focus on features that later could be expanded as conventional statistical concepts and representations. However, the focus and meaning attributed to the software representation varied from student to student: Whereas some of them, guided by Kay's question, emphasized the everyday consequences of their choices, others, perhaps as a result of previous experiences with decontextualized mathematical problem solving, seemed to give priority to the number manipulation aspects of the task.

The Meaning of the Data

Following Blake's proposal to use 80 hr as the cut [24–32], Kay asked the other students to present questions to him [33]. The students' reactions show another interesting contrast between their views on the overall meaning of the data displayed on the screen. The display shows results from a sample of the batteries of each type. Kay and those who planned the lessons may have assumed that the students were aware of this and looked at the data as a sample of possible outcomes. But Jamie's questions to Blake [34, 36, 39] suggest that this was not how she saw the data. Her questions suggest that she viewed the display as self-contained. In contrast, Blake's response [37] to Jamie, clarified with the help of Kay's comments [38], suggests that he saw the results displayed by the tool as mere samples.

Jessica and Sequoria, apparently trying to show that they understood Blake's point and that they agreed with him, commented on the consequences of choosing one of the low performing batteries "if you were using those batteries for something important" [42, 48]. But they only partially agreed with Blake: They acknowledged that his criterion was a better one if we consider the everyday consequences of choosing one or the other brand but, as in Jamie's case, they seemed to consider the display as self-contained, referring to "those batteries" [42] and not to a sample of all the batteries of that brand.

We have no information on what happened next and on whether Jamie changed her views. Her perspective demands careful analysis and follow up. As often happens to students who show difficulties in mathematics, the meaning and goals she attributed to the task do not seem to match the teacher's conceptions and the goals set by the teacher for the activity. Such disagreements often refer to subtle aspects of the activities that perhaps are taken for granted by the teacher and, as such, are not explicitly discussed in the classroom.

Refinement of Individuals' Meanings Through Social Interaction

Despite using the same representational tool and examining the same set of results displayed by the tool, the students approached the task in different ways, attributing different meanings to the displays. The two main sources of differences were the criteria to determine the better brand and the meaning of the data under discussion. Some students brought different and sometimes opposing perspectives to the discussion, whereas others refrained from the discussion altogether. Notice, however, that the individual points of view are intertwined, with more than one child contributing to each approach. Through discussion, the individual views were refined, becoming more explicit and consensual within a limited group. At the same time, within the limits of the videotaped episode I had access to, opposing points of view persisted. Moreover, even within the same approach, individual conceptions might have been slightly different.

The interactions and discussions did not seem to have a fruitful result when they took place between students with markedly conflicting views. For instance, the discussion between Blake and Jamie did not seem to change their perspectives. In other cases, some interesting cooperation patterns developed, as when Jamie rephrased Caesara's first statement or when Blake complemented Jason's suggestion to look at a different cut by implementing it with the tool and analyzing the consequences of this action. Blake and Jason focused on the same aspects of the distribution and seemed to be sharing the same meanings and goals. The same seemed to be true for Jamie and Caesara. As such, the interaction within each of these two pairs led to more refined and more advanced views, whereas the interaction between students with different views did not seem to allow for progress. In keeping with Piaget's (1965/1995) views, because the participants in the discussion did not share the same meanings, they could not cooperate, and one student's perspective could not become the object of reflection by the other. As long as this situation persisted, one could not expect progress in the student's views.

Let's now examine how extended common experience with the tool and with classroom discussion relates to the emergence of common views and of new mathematical understandings.

THE *AIDS* EPISODE

The series of discussions on different, conventional and nonconventional representation of data from the AIDS experiment (the 5 solutions or inscriptions in Figures 9–12 in the Appendix, the article by McClain, this issue) should clarify how experience with the software tools used throughout the classes contributed to the students' understanding of data representation. I argue, however, that more important than the experience with the tools were the types of questions students were asked to answer and the way the teachers intervened, rephrasing students' utterances and helping them to refine and to share their ideas, thus attempting to build a common ground for the establishment of a genuine discussion where cooperation and reflection on each other's views could take place.

In this class, held at the end of the teaching experiment, the students were asked to judge the adequacy of five solutions, consisting of conventional and nonconventional written representations showing the main trends in the data. The solutions differed in terms of whether emphasis was placed on visual, verbal, or numerical representation. The data on the blood cell count for 146 patients receiving a traditional treatment and for 46 patients receiving an experimental treatment would lead to contradictory conclusions depending on whether students compared absolute frequencies (the number of cases in a certain range) or relative frequencies (number of cases in a certain range in relation to the total number of cases in the group) across treatments. As conveyed by the teachers, a representation or solution was adequate if it provided a basis for deciding which treatment was better. As the class unfolded, the tension between the absolute and the relative size of frequencies became central.

Once more I focus on each student or group of students' main approaches as they analyze the adequacy of the solutions and as they compare relative frequencies. I try to detect instances where their ideas change and evolve through discussion with the other students and in response to the questions and comments by Kay, the researcher conducting the class, and by Paul, the researcher who collaborated with her during this class. I also try to evaluate how their approaches relate to their previous experiences with data displays provided by the tools.

From Individual Cases to General Tendencies

In the first assignment in the *AIDS* episode (Subepisodes 1 and 2), the students were faced with a schematic, nonconventional line graph solution (Figure 9 in the Appendix) showing the distribution of blood cell counts for the patients receiving the traditional treatment and for those receiving the experimental treatment. The students' task was to judge whether this was an adequate representation of the data if one wanted to choose a treatment for AIDS.

As discussion about the first solution began, 5 students, although agreeing that the display showed a good data representation, interpreted the graph in slightly different ways. Jamie stated that "you can see where the range is starting and ending and you can see where the majority of the numbers are" [52]. As Derrick questioned her on what she meant by "the majority of numbers" [57], Sheena explained, "When she says where the majority of the numbers were, it's like where the point is, you see where it goes up?" [62]. A plausible, although not testable, hypothesis is that this student's emphasis on the peak in the curve resulted from her experience with the software tools where there was an overwhelming focus on lengths and heights. Sheena may have considered the peak as corresponding to the bar or the column with the highest frequency in the histogram-like display provided by the tools. But this does not seem to be what Jamie had in mind. Jamie mentioned the range starting and ending [52]. She seemed to be considering all the data under a curve. Vallory agreed with Jamie as she stated, "However many people were tested, that's where most of those people fitted in, in between that range" [72]. Notice, however, that she considered the relative nature of the data as she referred to an unspecified number of cases ("however many people were tested"), an aspect not visible as a physical feature of the displayed solution and apparently not contemplated by Sheena or by Jamie.

Kay coordinated the discussion, made sure the students listened to each other, and raised questions or rephrased students' utterances. Her remarks seem to help the students refine their views. For example, when Kay questioned Derrick's expression "the highest range of the numbers," Sheena, who had first agreed with Derrick, seemed to change her mind as she exclaimed, "Oh, no" [66–69].

In this interaction, the students cooperated insofar as they considered each other's views, gradually reaching a more complete view of the display's features. Although they reached no final agreement, the discussion led to a more precise description of what the graph meant, revealing a true collaboration between students in specific groups and between students and teachers. Without dismissing Jamie's first description, Vallory came up with a more precise statement for the same interpretation, extending it to include reference to the relative nature of the numerical information. Paul intervened at the end of the subepisode to give a precise account of what Vallory meant by "that range," pointing to the beginning and ending points of the graph depicting the data on the old treatment [73].

One of Vallory's contributions, however, was not followed up either by the other students or by the teacher–researchers. When she referred to the majority of cases in relation to "However many people were tested" [72], Vallory seemed to be dealing with the main issue to be later discussed in this class, namely, the relative character of the compared frequencies in the display. Whereas Jamie, as she did in the analysis of the batteries data, seemed to focus on the cases instantiated by the graph, Vallory looked at the cases as a description of general relations. In this first subepisode, neither Kay nor Paul pursued Vallory's comment and did not try to engage the students in discussing the relative character of the trends displayed in each graph. It was only later (Subepisode 5) that relative frequencies were explicitly proposed as an important issue. It is not clear why Vallory's point was ignored. Perhaps the teacher–researcher team did not expect the students to focus on relative frequencies before the discussion of Solutions 2 and 3 (see Figure 10 in the Appendix and Subepisode 3), and, for that reason, Kay and Paul did not recognize Vallory's comment as a first step toward that discussion.

As in the *Batteries* episode, students approached Solution 1 in the *AIDS* episode from different individual perspectives, taking into account different features of the representation under discussion. Sheena seemed to focus on the shape of the graph as it related to specific points previously represented by the software tools as one bar or as one column of points. Jamie looked at the range and the specific numbers that fell within that range and seemed to be using an approach similar to what she used in the *Batteries* episode, when she focused on the data displayed and not on the general pattern exemplified by them. Vallory, instead, seemed to look at the pattern as a general property that could be applied to any set of cases. These different perspectives regarding Solution 1 are very likely to have emerged from their previous experiences with the tools. However, it was through interaction and discussion, as students and teachers took into account each other's perspectives, that progress took place and certain features of the representation came to be recognized by students who, initially, did not seem to be aware of them.

Mathematical Relations (Almost)

At the beginning of Subepisode 3, Kay showed Solutions 2 and 3 (Figure 10 in the Appendix) and asked the students to analyze the written statement in Solution 2: "The new drug was better than the old. The majority of the old ones are behind 550, and the majority of the new drug was in front of 550" [75]. The teacher–researchers' choice of Solutions 2 and 3 seemed to be part of a plan to lead the students to progressively focus on relative frequencies and multiplicative relations as the basis of their evaluation of data distributions.

The students, however, started by focusing on certain features of the display that apparently were emphasized in previous lessons. Vallory asked why 550 and not the median was chosen as the cutoff point [78]. The children may have worked before with the concept of median and may have used the tools to determine the point that separates the number of cases into two equal parts. We saw in the *Batteries* episode that this was an idea brought into the classroom discussion by Jamie and Caesara.

Vallory's question triggered a series of statements by Kiri, Kay, Megan, and Paul that progressively introduced and defined the idea of a natural break in a frequency distribution. Kiri proposed that "550 might not be the middle of the data, but it's the middle of whole scale" [80]. Kay rephrased Kiri's idea as "the middle of the range" [81]. Megan mentioned the appearance of the distribution when the software tool was used [88], but her statement concerning how the data were distributed in relation to 550 was not clear. This allowed the teachers to smoothly bring in the idea of natural breaks in the frequency distribution: Kay referred to 550 as a natural break in the data [89]. Paul corroborated the notion of a natural break by describing what he saw happening in a previous discussion by one of the groups. Pointing to the curves in the display discussed in the previous section (Subepisode 2), he defined a natural break as "the place sort of between the hills" or "where this kind of hill started and where this kind of hill finished" [90].

The notion of a natural break as a flexible point that divided the cases under discussion into two sets was explored further as Kay directed the students to discuss Solution 3. She commented that "this group did a similar thing because they said they looked at how many of the T-cell counts were between 200 and 525, and they looked at how many were between 525 and above. So they used 525, and these people used 550" [91]. But this third display included information on the number of cases below and above 525 for each of the two groups. This numerical information immediately triggered Marissa's comment that this new description was "more confusing" given that one program had "more numbers" than the other [92]. Marissa's timely comment allowed Kay to direct the students' attention to the values in the display. She expanded on Marissa's utterance and stressed that, in the old treatment, "they had 56 that were in this upper range which is where we want to be and these [the new program] only had 37 [in the upper range]" [95]. She then paved the way for students to discuss the values in relation to the total amount of cases in each group. She did so by challenging them with a statement that contradicted what they apparently knew to be true: "So somebody might say the old program was better because there were more [successful results]." Jamie accepted Kay's challenge and expressed her view concerning the relative nature of the numbers: "This is related. You know like, like a, like a scale drawing where it's not the same size, but it's the same thing."

These interactions show possible influence from the students' prior experiences with the statistical software tools. The students brought to the discussion the concepts of median and range, which probably were dealt with before and demonstrated in displays produced by the tools. Another concept that could have been generated by experience with the tools is that of a natural break. In the preceding whole group discussion, however, it was Paul, the researcher participating in the discussion, who mentioned it as part of his description of what a group of students had done before.

The whole episode constitutes another example of cooperation between the teacher and the students toward refinement of their initial analysis, in this case, of Solutions 2 and 3. This cooperation seems to be possible because, as the participants discussed the displayed solutions, they could follow each other's questions and meanings and, building from them, start focusing on more complex mathematical relations.

Kay used the apparent contradiction pointed out by Marissa to challenge the students. This allowed Jamie to move forward in her way of looking at the data. She then referred to the relative character of the frequencies in question, a feature not included in her previous utterances in the video-taped subepisodes that we have access to. Her wording, although not very clear ("it's not the same size, but it's the same thing" [96]), deals with the core issues that are discussed explicitly next. Unfortunately, the selected videos do not allow us to examine how Marissa's ideas about Solution 3 evolved in the ensuing discussion.

The Relative Character of Frequencies

The students were directed further to explore the relative character of frequencies in Subepisode 4, as Paul revived Kay's challenge, now in relation to Solution 4 (Figure 11 in the Appendix), a new display involving the same numbers as the previous one. He did so by arguing that the greater (absolute) number of cases with T-cell counts above 525 in the old treatment group signified that the old treatment was better [99]. Blake and Jason were the first to respond, initiating a new round of interactions leading to refinement and expansion of the students' views regarding the relative character of the frequencies under discussion, an issue first raised by Vallory in Subepisode 2 and later by Jamie in Subepisode 3.

In his immediate reaction to Paul's challenge, Blake pointed out, "there's more people with the old program than there is with the new program" [101]. Jason extended Blake's comment by including specific information on how the displayed frequencies related to the total number of cases in each group: "37 is more than half over 525 and 56 is not more than half of 130…more of them on the bottom than on the top." [103]. As happened before with other student pairs or trios, one student, Jason, expanded on the relations mentioned by a peer (Blake) showing how they applied to the specific frequencies at hand.

Paul and Kay encouraged the other students in this promising discussion [105, 106]. This gave Jason the opportunity to further develop his analysis: When Derrick complained that he did not hear what was said, Jason repeated his description of the display, improving it in the process by relating it to the ideas of "more than half" and "less than half" of the number of cases in each group: "OK, you see how 37 is more than half of 9 and 37 together? But 56 is not more than half of 30, 130, and 56 put together. There's more on the bottom one than on the top one" [112].

Kay asked Will to repeat Jason's explanation in "a different way " [113]. Will explained

> On the bottom one you have, see what Jason was saying there's more than there is below 525 and so that means that that one is better because the top one it doesn't even have close to half of what the one below 525 is on that one. So that means that if, if that was the same amount of people it had like, if they both had the same amount of people and, but, and they had the numbers and everything, and this one, the bottom one was a however much more that of…." [116]

Will made the comparison more comprehensible by considering "if they both had the same amount of people." He did not complete his argument but seemed to adopt a perspective that was somewhat equivalent to Vallory's point at the end of Subepisode 2, suggesting awareness of a general mathematical relation that could hold for samples of different sizes [72]. The subepisode ended when Derrick, who apparently had fallen asleep, hesitantly expressed his understanding of Jason's analysis: "He's saying that, that half, like 37 is over half of 526, 3, 5, 6, uh, uh, 56. And uh, like if it was more people than it would probably be higher than, than 6 like…" [120]. Although Will and Derrick did not fully express what Jason had explained before, their participation in the discussion may have helped them to start considering the relative character of the frequencies under discussion.

In this subepisode, the students responded to the teachers' challenges by taking turns in describing, with different levels of precision, different features of the data display. As they explored the number relations in the display, they went beyond the data in the display and showed understanding of basic logico–mathematical properties that, as far as shown in the video-taped subepisodes, were not explicitly presented either by the software tools or by the solution displays. Experience with data distributions presented and organized by the software tools may have provided the students and the teachers with the context and the common referents for the discussions about mathematical relations. However, the available video clips contain no evidence that the students' understanding about the relative character of frequencies emerged from use of the tools. The students' understanding about relative frequencies seemed rather to emerge from their individual reflections about the general trends and numerical relations in the data included in Solutions 1 through 4, as they participated in the discussions and tried to answer the teachers' and the other students' challenges.

Numerical Information Versus Mathematical Relations

As the students discussed Solution 5 (Figure 12 in the Appendix), the focus of the last subepisode, some further developed their views on the relative character of frequencies and expressed them as general statements. Solution 5 corresponds to a quartile distribution where, for each treatment group, the line showing the whole range of results from 220 to 850 is divided into the four parts that each contain one fourth of the cases. The assignment proposed by Kay, similar to the previous ones, was "to see if you can understand what these folks did, and then decide if you think it's adequate" [122].

Blake volunteered to explain his point of view. He thought the graph was adequate and mentioned that "the numbers were the same on this" and that "they split into four groups" [127]. From what is shown in the clip, it is not clear how Blake immediately found out that the four parts in each display contained equal number of cases. His overall statement suggests that he could relate the new graph to other graphs showing where the individual cases were placed along the range, as typically happens in the displays created by the software tools used throughout the classes. He seemed to be contrasting this new representation where the frequencies were equal to a representation where, instead, the ranges would be equal: "like you know when people use the range and stuff but the numbers were the same on this" [127]. Here Kay, probably comparing the displays for each treatment, commented, "It's helpful that the ranges are the same" [128]. Blake didn't explicitly acknowledge that the equal ranges would matter [129] and concluded that the new treatment worked better than the old treatment "because the three lines for the equal groups were all above 525 compared to only one of them was over on the top"

[135, 137]. Note that he referred to three lines, the prominent visual feature in the display, when he probably meant the three spaces between four lines, or the cases contained in these three spaces.

Marcus commented on Blake's view but suggested that "it might have helped if they put the numbers in the groups so you have a better idea of what you're seeing" [139]. Kay and Paul followed Marcus suggestion and started discussing how one could find out the numbers that would go into each quartile. Marcus worked out some numbers while Derrick said that they could not determine what the numbers were. Blake, however, maintained his original view and interrupted the discussion, appropriately stating that "it doesn't really matter where the all the data is because you know from where the groups are what, what treatment is better or where the data stands on both treatments" [152].

The subepisode ended with Blake's evaluation of the graphical representation. His main point was similar to what Vallory had proposed at the end of Subepisode 2: "However many people were tested, that's where most of those people fitted in, in between that range" [72]. In other words, what mattered were not the absolute but the relative values of the frequencies displayed.

Although some children brought up features of the tools in a more restrictive way, Blake seemed to use it to reflect on the general characteristics of the data distribution. In the batteries discussion he showed that this was his typical approach, and therefore it was not clear whether his final achievement resulted from experience with the tools or from earlier experiences. His approach went beyond what was depicted in the display. Other students (e.g., Marcus), even at the end of the AIDS discussion, seemed to be focusing on numbers per se and on computational activity. His use of numbers and of computations exemplifies how mathematical tools and representations per se can be seen by students as sufficient in mathematical discussions, preventing them from further exploring more general mathematical properties. The most interesting cases are those of students (e.g., Jamie) who showed progress in their views as they participated in this series of discussions.

In all these cases, experience with the tools may have provided the context for students' reflections on number relations. But the conclusions they reached concerning the analysis of relative frequencies were triggered by their own individual approaches and by the classroom discussions and questions they were asked to answer. Regardless of their level of complexity, the students' views represent further mental elaboration on what is presented by the tools or by the different displays or solutions.

SOME FINAL CONSIDERATIONS

My answer to the main question dealt within this volume is that the tools used in the classes did not constrain the students' interpretations of data distributions or the way they reasoned about frequencies. The video episodes show how the data display tools and the graphic representation of solutions used in the stats project classes provided a context for discussions about data trends, helping students to make decisions based on data distributions and to consider the relative character of the frequencies they were asked to compare.

In each of the episodes we saw individual students approaching data displays and the tasks proposed by the teacher from different individual perspectives. As I have argued throughout my analysis, the students' comments about relative frequencies seemed rather to emerge from their individual reflections about the general trends in the displays and solutions they were asked to consider, as they participated in classroom discussions and tried to answer the teachers' and the other students' challenges. Such discussions, I believe, could have occurred and led to similar results, in terms of understanding relative frequencies, even if students had not previously worked with the software display tools. This hypothesis, although not testable within the confines of the available video data, is reasonable if we consider how children, adolescents, and adults apply proportionality principles as a tool to solve problems in a variety of contexts where no organized data display tools are offered to the problem solver (see, e.g., Inhelder & Piaget, 1958; Piaget, Grize, Szeminska, & Bang, 1968/1977; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990; Schliemann & Magalhães, 1990).

Students' individual approaches in the classes may have benefitted from their work with the software tools, but they were also the result of the students' previous experiences in mathematics classrooms, the meaning and values that students attributed to the different features of the displays they discussed, the teachers' questions, and the evolving interaction with their peers. Some students focused on the visual features of the tools and the specific numbers in the displayed distributions. Others used the tools and the nonconventional data displays as opportunities to reflect on general characteristics and properties of data distribution and of mathematical relations that transcended the specific displays.

The two opposing views on the meaning of the batteries data—self-contained set versus data sample—are somewhat related to the often unresolved issues that students face in their understanding of algebra, as opposed to arithmetic. Children struggle to solve problems involving variables and general mathematical relations, as opposed to problems with unknowns (e. g., Carraher, Brizuela, & Schliemann, 2000). Arithmetic problems refer to unknown values or to specific instances of a relation, and, as such, are self-contained, as the interpretation adopted by Jamie was self-contained. In algebra classes, students are asked to deal with variables focusing on general properties of number relations that apply to an infinite number of possible sets and not just to those displayed in a graph or in a table of measured values. In a way, this is akin to the way Blake looked at the data as an instantiation of an infinite number of possible samples.

The tension between the particular and the general character of representations and mathematical relations also came into play in the analysis of the AIDS treatment data. Some students, like Marcus, focused on the computation of specific numbers. Others, like Blake, focused on the structure of the mathematical relations implicit in the frequency distribution displays. In Sfard and Linchevski's (1994) terms, we can say that Marcus looked at the task as a process, whereas Blake treated the displays as mathematical objects or structures.

As the different approaches evolved and changed, the complementary nature of individual and sociocultural contributions to reasoning processes became evident. As students were introduced to new tools and symbolic systems, the teacher played a crucial role in guiding and shaping classroom interactions. However, not all interactions were sources of development of mathematical reasoning. For effective interaction and cooperation to take place, participants must use similar systems of signs and meanings and carry out similar operations (Piaget, 1965/1995). Here is where data displays and graphic representations help maintain common referents for the discussions. But the meanings that individual students attributed to the displays may have limited how far they explored and conceived of the implicit mathematical relations under discussion.

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