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The Interplay of Intimations and Implementations: Generating New Discourse With New Symbolic Tools

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The analysis of the *Batteries* and *AIDS* episodes presented in this article is guided by the assumption that thinking can be conceptualized as an activity of communication, and learning can be regarded as modifying and extending one's discursive ways. Within this framework, 1 of the aims of mathematical learning is to become skillful in the discursive use of designated symbolic artifacts supposed to mediate solving certain types of problems. My analysis of the learning episodes is aimed at uncovering the ways in which the discursive uses of such new symbolic tools were interactively constructed by the students. I argue that the construction process is extremely complex because of a certain inherent circularity of this process. The analysis reveals that this difficulty may be overcome by the gradual dialectic adjustment of former discursive habits to new contexts. The adjustment happens gradually, through cycles of *intimations* about the applicability of the old habits followed by *implementations* in which this applicability is examined. This intricate interplay of intimations and implementations is found in both the *Batteries* and the *AIDS* episodes and is presented in detail in the article. I also show that, with time, students significantly increased their mastery of this particular discursive mechanism.

Like all the contributors to this issue, I analyze the *Batteries* and the *AIDS* episodes and, like everyone else, I focus on what was described in the invitation to this special issue as "the ways in which symbolic tools and other artifacts enable, mediate, and shape mathematical thinking, while being themselves, at least to some extent, a product of these processes." And yet, although the general goals of my article may be similar to those of the other contributors, my conceptualization of basic con-

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cepts, such as mathematical thinking, is probably quite different. This, of course, leads to a distinct type of analysis.

Even before I introduce my conceptual infrastructure, let me specify the questions that I am trying to answer in this article. In most general terms, my aim is to fathom the nature and mechanism of the special type of learning that took place in the stats project classroom. This learning can be described as aimed at making the student able to use graphic displays of data in solving preference problems. The last term, *preference problems,* refers to the type of problem that the stats project class grappled with in both the *Batteries* and the *AIDS* episodes. The defining feature of this type of problem is that it requires making a choice between options represented by sets of numerical data. The similarity of the tasks and of the tools available in the *Batteries* and the *AIDS* episodes implies a possibility of similar solution and, in particular, of similar uses of the different symbolic artifacts. In situation like this, the question about learning seems justified. Because the two instructional tasks resemble each other in so many ways, it is only natural to ask whether students' performance in the latter episode can count as in any way improved compared with their performance in the former.

Of course, the preliminary question is what change should count as improvement. The principal task of the problem solver is to decide how to look at the available display of data. In preference problems, the way of looking is considered as adequate if it makes it possible to determine the preferable option. From the *Batteries* and the *AIDS* episodes it is quite obvious that constructing a way of looking is not a trivial matter. The pervasive difficulty of the task is well known to mathematics teachers, who, only too often, observe their students grappling with—and failing in—scanning different kinds of graphs for different kinds of information. Seeing things in displays is not a matter of just looking. What cardiologists can see in electrocardiograms and what architects notice in blueprints often remain invisible to the layperson. This means that seeing what is regarded as relevant for a given problem requires learning. This and other intriguing phenomena related to constructing the ways of looking at symbolic artifacts have been neglected for too long. Recently, the situation has begun to change, and the issue of constructing the ways of looking is gradually gaining the attention it deserves (see, e.g., Goodwin, 1994; Goodwin & Goodwin, 1996; Goodwin & Ueno, 2000; Roth & McGinn, 1998; Stevens & Hall, 1998).

All these examples indicate that symbolic artifacts do not speak for themselves. Even our ability to notice certain elements, or to see several of them as the same or different, depends on the way in which we attend to visual displays and scan their details. The attending procedures, in turn, are functions of our goals and of what, in a given community, counts as a proper way of realizing these goals. Thus, my first aim in this article is to specify adequate ways of looking at symbolic displays such as the bar diagram in the *Batteries* episode (see Figure 1 in the Appendix, the article by McClain, this issue) and the dot plot in the *AIDS* episode (see Figures 3 and

5 in the Appendix). The term *adequate* refers to scanning procedures that lead to solutions that stand a good chance of being adopted by their prospective users (battery buyers and medical doctors, in our case).[1] The first question I try to answer in this article thus may be formulated as follows:

> *Question 1*: What can count as the adequate ways of attending to data displays such as those in the Batteries and AIDS problems?

Whereas what is needed to answer this query is a careful analysis of the problems and their possible solutions, my other two questions require looking at what the students actually did. With the help of our data, I try to fathom the mechanisms that underlie construction of these special ways of attending to symbolic displays. Thus, my second question is as follows:

> *Question 2*: How do problem solvers construct adequate ways of attending to data displays?

My final aim is to see how much learning took place in between the two episodes:

> *Question 3*: How much learning occurred between *Batteries* and the *AIDS* episodes? More specifically, how skillful did the stats project children become in constructing adequate ways of attending to data displays?

Before I address these three goals, however, I have to present the conceptual framework that guides this effort.

CONCEPTUAL PROLEGOMENA: COMMUNICATIONAL VIEW ON THE USE OF SYMBOLIC TOOLS IN MATHEMATICAL PROBLEM SOLVING

Thinking As Communicating and Learning As Initiation to a Discourse

The batteries bar diagram and the AIDS dot plot are symbolic tools, and, as any such artifact, they have been designated to serve as means of communication. This fact must be kept in mind throughout the analysis. After all, whatever is being done with the symbolic tools is a result of the users' attempt to agree on, and improve, these artifacts' communicative function.

[1]One may assume that what is found truly helpful by the users also would be deemed as appropriate by the community of statisticians.

This last statement makes it clear that this analytic endeavor can be described as dealing with the issue of communication, that is, with the question of how people coordinate their actions in the pursuit of agreed goals. On the face of it, focusing on communication is a deviation from the stated intention to investigate "the ways in which symbolic tools and other artifacts enable, mediate, and shape mathematical thinking." In fact, it is not necessarily so. Within the conceptual framework I now introduce, thinking is regarded as a special case of communicative activity (Sfard, 2000a, 2000b). Indeed, when one comes to think about it, a person who thinks is communicating with herself. Our thinking is clearly a dialogical endeavor,[2] where we inform ourselves, argue, ask questions, and wait for our own response. If, in addition, we keep in mind that communication does not have to be verbal, equating thinking with communicating becomes even more plausible.

Let me add a number of conceptual clarifications. The word *communication*, around which the whole framework evolves, can be found in literature defined in many different ways. In this article, I regard communication as an activity in which one is trying to make an interlocutor (possibly oneself) act or feel in a certain way. This definition, with its roots in the contemporary pragmatics (e.g., Grice, 1975; Levinson, 1983) has been shown to be more operational than the typical dictionary descriptions, especially those that feature expressions such as "exchange of meaning or feelings" or "transfer of information" (see Sfard, 2000a). It is important, however, to stress that some discursive psychologists (Edwards, 1997; Edwards & Potter, 1992; Harre & Gillett, 1994), whose approach to cognition is not unlike the one I called communicational, would hesitate to adopt the pragmatist definition of communication. Following the lead of Ludwig Wittgenstein (1953), they would maintain that the only way to prevent conceptual circularities in psychology is to purge the discourse on human thinking from any mention of mental entities supposed to underlie human actions. The pragmatist definition of communication, which implicitly refers to intentions, seems to disobey this rule. However, because conceptual entanglements arise only as a result of attempts to compare feelings, meanings, or intentions of different people, and because no such comparisons are required following the pragmatist definition of communication, it seems that we may use this definition safely, without fearing the dangers of conceptual entanglements that Wittgenstein was worried about.

Within this framework, the word *discourse* is used to denote any specific act of communication, whether verbal or not, whether with others or with oneself, whether synchronic (like in a face-to-face conversation) or asynchronous (like in an exchange of letters or in reading a book). This certainly is a very broad meaning, and, as a result, the notion of discourse used here encompasses more types of communicative activities than allowed within the colloquial use of the term. To com-

[2]Compare Mikhail Bakhtin's dialogical vision of mind (Bakhtin, 1986).

plicate matters even further, the same word may be found in the literature as referring to whole families of discursive acts that display common characteristics and that can be identified with groups of people who do not necessarily know each other but are united by a common goal and prepared to act according to shared principles.[3] Which of these two different meanings of the word *discourse* is meant is usually easily understandable from the context within which it appears.

It is this latter meaning of the term *discourse* that one has in mind while describing it with such adjectives as *mathematical*, *statistical*,[4] *historical*, or *everyday*. Although discourses are dynamic, ever-changing entities with many overlaps and with numerous irregularities, one can still distinguish "discursive formations" (Foucault, 1972) that are distinct enough and sufficiently stable to justify the talk about different kinds of discourses. If so, and if we regard thinking as communicating, the term *discourse* may become a substitute for *knowledge*,[5] and the notion of learning can be redefined to denote the activity of becoming a skillful participant of a certain specialized type of discourse. Thus, learning mathematics in school becomes an initiation to the special discursive practices known as school mathematical discourse. The use of the word *school* in this last sentence is meant to stress the substantial difference between discourses of professional communities and those that develop under the guidance of schoolteachers (cf. Rittenhouse, 1998). Pedagogical needs and constraints of institutional settings have a major impact on the nature of school discourses and on the ways they develop.[6]

Mediating tools that enable communication are among the most important criteria for distinguishing between discourses.[7] For example, symbolic artifacts typically used in mathematical discourse are quite unlike the visually accessible mediators used in everyday discourse. In this latter discourse, interlocutors seeking to ensure effectiveness of communication often would have recourse to everyday material object. They would point to, look at, or just imagine the things they

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[3]Such groups are known in the professional language as *communities* or *communities of practice*. For a more elaborate treatment of this last concept, see Lave and Wenger (1991) and Wenger (1998). The idea of discourse as the communicative activity typical of a certain community is closely related to several other notions that can be found in literature of discourses. See, for example, *discursive formation* (Foucault, 1972), *Discourse-with-the-capital-D* (Gee, 1997), *genre* (Bakhtin, 1986; Lemke, 1993), and *register* (Halliday, 1978; Halliday & Martin, 1993).

[4]In this article, I use the terms *statistical* and *mathematical* interchangeably because the two discourses described with these adjectives share all the characteristics that are considered in my analysis.

[5]Foucault (1972) might have been one of the first writers whose work implied this conceptual shift. The conceptual nuances of the issues of knowledge, power, and communication have been dealt with in a similar spirit by the Scandinavian school initiated by Rommevtveit (Rommevtveit & Blakar, 1979).

[6]Compare the notion of "didactic transposition" (Chevallard, 1990).

[7]See also Foucault's discussion (1972) on criteria for distinguishing between different discursive formations.

are talking or thinking about. In advanced mathematical discourse, the only visual mediators are multifarious, alpha-numeral, and pictorial inscriptions.

Another distinctive feature of discourses is *metadiscursive rules*, that is, mostly tacit navigational principles that seem to underlie any discursive decision of the interlocutors. It is thanks to these rules that the speakers know "*when* to do what and *how* to do it" (Bauersfeld, 1993; cf. Cazden, 1988). There are many general metarules common to all kinds of communication. In addition, any particular discourse is subject to more specific regulations that work for this discourse and do not hold anywhere else. Mathematical discourses imply, among others, unique ways of attending to inscriptions, because of which we are prone to notice some aspects of these inscriptions while remaining unaware of others, even if these other aspects are pretty obvious to participants of another discourse. In mathematical discourse, this is what underlies, for instance, our instinctive decisions to attend to the degree of a variable in any algebraic expression and to ignore other features, such as the shape of the letters in which the expression is written. These discourse-specific metarules are responsible for the uniquely mathematical ways of viewing situations as the same or different; they also underlie the decisions about what kinds of perceptual mediators count as "representations of the same mathematical objects," that is, are fully interchangeable in the discourse. Yet another set of metarules enables us to decide whether a given solution can be regarded as complete or whether the given argument can count as final. In the mathematical discourse, whether professional or the one that develops in school, the criteria of acceptability, adequacy, and equivalence are much more restrictive and more rigorous than in everyday discourses (for a more detailed treatment of mathematical metarules, see Sfard, 2000b).

Let me conclude these introductory remarks with a few words about the way in which the communicational approach affects educational research. First, because learning mathematics is conceptualized as developing a discourse, probably the most natural unit of analysis in the kind of study I undertake in this article is the discourse itself. More specifically, my focus here is on the discourse generated by the problem solvers around the new mediating tools. While discussing the development of this special type of communication, I "zoom" in and out, alternating between analysis of students' single turns and the examination of whole sequences of thematically connected utterances. This may be compared with the study of the mechanics of water where, at some points, the researcher may wish to investigate the geometry and periodic recurrence of waves and whirls, and at other times she may choose to watch the movement of individual particles, shaped by forces of their mutual attraction. The macro- and microlevel pictures obtained in these ways do not resemble each other, but both are needed to understand the complex phenomenon under study. Similarly, in the analysis that follows, whatever my particular focus, the main question remains the same: What is the nature of the discourse that develops, and what mechanism underlies this development?

Second, the ontic/epistemic specificity of my unit of analysis is also the reason why certain modifications of our vocabulary may be necessary. Along the next pages, I try to keep the number of terminological innovations to minimum. And yet, I cannot avoid them altogether. Although, very often, new terms seem almost tantamount to certain familiar notions, these latter notions may be too loaded with old ontic and epistemic connotations to be compatible with the new conceptualization. In the particular case of communicational approach, in which the object of analysis is discourse, I try to avoid terms that have, so far, been used within a traditional cognitivist framework and may thus carry with them undesirable "acquisitionist" entailments, that is, tacit references to the entities "acquired" by the learner and stored in her mind (Sfard, 1998).

Third, as the metaphor of the study of water makes clear, by defining thinking as communicating we may have sidestepped the famous split between individual and social research perspectives. As many writers have noted, this split results in two incompatible and somehow incomplete types of studies (each of these frameworks has been accused of "telling only half of the good story"; see, e.g., Cobb, 1996; Confrey, 1994; Lerman, 1996). The problematic division is no longer an issue when one realizes that the cognitivist and interactionist approaches are but two ways of looking at what is basically the same phenomenon: The phenomenon of communication, one that originates between people and does not exist without the collective, even if it may temporarily involve only one interlocutor.

Turning Inscriptions Into Communication Mediators: Constructing Discursive Focus

Once we define learning mathematics as an initiation to a certain type of discourse, it becomes clear that such learning involves becoming adept in specifically mathematical ways of communicating. This learning thus must lead to special uses of mathematical symbols and to uniquely mathematical metarules. Like in everyday discourse, where the familiarity with the objects of conversation helps interlocutors sustain coherence and mutual understanding, so in mathematics, a good sense of designated symbolic tools and of their relevant features is the basis for creating a clear discursive focus.

Because of its significance for my analyses, the term *discursive focus* requires clarification (cf. Sfard, 2000a; Sfard & Kieran, in press). There seems to be more than one way to define this concept. We may use the term *focus* to denote the words used by an interlocutor to signal what he or she is talking about. This type of focus is called *pronounced*. Thus, if we are looking at the graph in Figure 1 and say, for example, "The value of this function for $x = 3$ is 2.5," the pronounced focus is the expression "the value of this function for $x = 3$." Alternatively, we may decide that the term refers to what and how we are attending to—looking at, listening to, and so forth—when speaking. In our example, this type of focus, which I call *attended*, may involve the following attending (scanning) procedure:

1. Find point $x = 3$ at the x-axis.
2. Erect (imaginary) vertical line at $x = 3$.
3. Draw (imaginary) horizontal line through the intersection of the line drawn in (*b*) and the graph.
4. Find the intersection of the horizontal line drawn in (*c*) with the y-axis.

It is clear that the information about the attending procedure that should be used in the given context is not inscribed in the graph itself. Rather, it is something that we have learned throughout our former participation in mathematical discourse. If we ever know how to use any symbolic display as communication mediator, it is because the words that have been pronounced evoke, in the context in which we hear them, a whole cluster of past experiences with this type of symbolic artifact and relate us to an assortment of discursive decisions that, according to our former practices, seem appropriate in this context. The set of experiences and discursive potentials associated by an interlocutor with the given pronounced focus are called the *intended focus* of this interlocutor.

Using this terminology, I can now say that conversation participants see their communicative efforts as effective if they have a good sense of their own discursive foci, can interpret the foci of their partners, and believe that there is full compatibility, if not simply identity, between the two. Similarly, an observer who tries to assess an attempt at communicating will see this attempt as effective if, according to her interpretation of the interlocutors' discursive foci, these foci are well matched with each other (well matched means either the same[8] or related in well-defined, appropriate ways). In situation like this, the participants and the observers are all likely to say that "everybody is talking about the same thing."

[8]Note that the words *the same* and *identical*, or *different*, for that matter, do not signal direct comparisons between foci belonging to different interlocutors; rather, they refer to comparisons made by the listener between her own foci evoked by the interlocutors' utterances. At this point, it is important to stress the interpretive nature of the observer's effort. Just like the more active discourse participants, the observer is involved in constructing attended and intended foci that fit with what has been said by the interlocutors. Unlike the latter, the observer who is looking at records of actual conversations has an opportunity to try to bracket her own spontaneous understandings and to look for other plausible, if unfamiliar, interpretations. One of her principal aims will be to reconstruct speakers' attended foci, which are their ways of scanning inscriptions. Aware of the ever-tentative status of her interpretations, the analyst will consider any scanning procedure that makes sense in the view of things said by the interlocutors. In this interpretive effort, she will use as evidence interlocutors' direct statements on their ways of scanning and, in the absence of the latter, she will look for procedures that are likely to produce the claims made as the result of the scanning.

Within this context, the importance of well-defined attended foci for the effectiveness of communication cannot be overemphasized. Not only is the attended focus "the public exponent" of the inherently private intended focus, but it also can count as the operative version of the latter. The attended focus, by telling us what to do, helps us in making discursive decisions. In the two episodes I analyze here, knowing the focus of the conversation will express itself in having a good sense of what is being attended to and what is being intended when words are used such as, for example, *a consistent battery* in the first episode and *more effective treatment* in the second one.

The requirements with respect to mathematical attended foci are quite unlike those in any other discourse. As exemplified with the case of the function presented in Figure 1, a mathematical attended focus often would be algorithmic, and even if not, it must be built according to rather rigid metadiscursive rules. Similarly, it seems reasonable that in preference problems, attended foci should fulfill a number of requirements to be accepted by the users as a helpful solution of their problem. What can count as an "appropriate" focus in the context of preference problem is discussed later. For now, I limit myself to showing the result of focal analysis as applied to a single utterance in the *AIDS* episode:

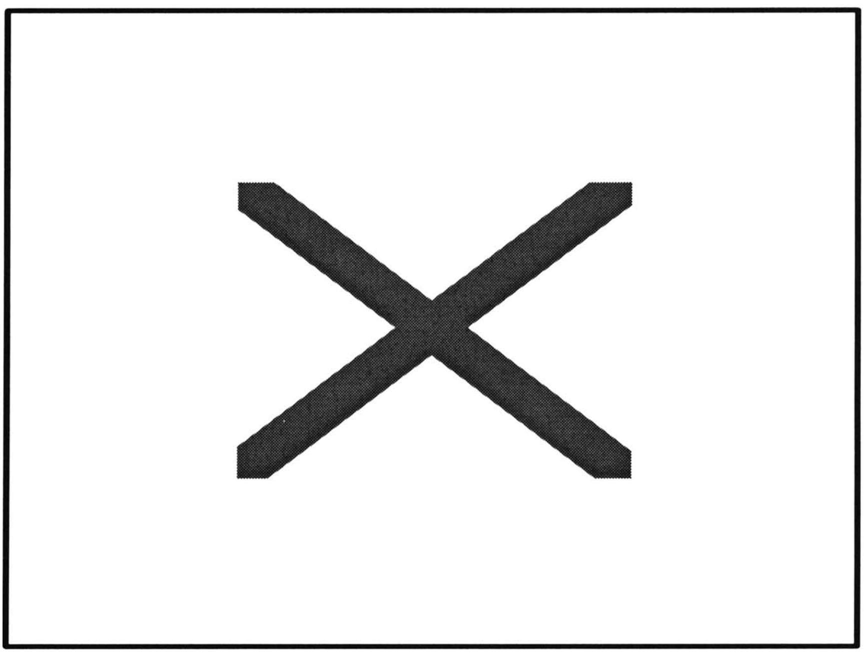

FIGURE 1 Scanning the graph of function to find its value for $x = 3$.

56. Jamie: OK, I think that it's an adequate way of showing the information because you can see where the range is started and ended and you can see where the majority of the numbers were [56].

This utterance seems to imply three-partite discursive focus presented in Figure 2, (this is the case at least in the last part of the sentence):

This focus is later deemed by the class as falling short of what may be expected in solving the problem at hand. The dissatisfaction with what is available at the given moment in the discourse is the driving force behind further development of this discourse.

Interplay of Intimations and Implementations as the Mechanism That Underlies Construction of a New Discursive Focus

Learning to solve preference problems now may be described as becoming skillful in a discourse in which effective attending procedures routinely are constructed and then applied as a basis for decision making. At a closer look, what is to be learned is far from simple. There is an inherent difficulty in trying to build the required procedures. The number of possible ways of scanning a symbolic display is infinite. The choice of one of these ways can be made only under the guidance of an intended focus. The latter requires a crystallized sense of what it means for an option to be better, and here is where the difficulty begins. In the *Batteries* episode, the children may well be aware that the longer the life span of a battery they bought, the better they spent their money; in the *AIDS* episode, they may know that the higher the number of T-cells, the greater the patient's chance for survival. None of these pieces of knowledge, however, translate automatically into a clear-cut attended or intended focus. Indeed, enough to look at Jamie's utterance [56] to see how vague and unhelpful the initial intended focus may be. A good sense of what may be called the quality of treatment, clear enough to lead to a firm preference, can arise only from a mathematically adequate, fully effective, attended focus. It seems, therefore, that there is a mind-boggling circularity here: Finding an effective attending

Pronounced focus	Attended focus [attending procedure]	Intended focus
"where the majority of numbers are"	[Find] the interval on x-axis corresponding to the protruding and dense aggregation of points	The aspect that shows the quality of treatment -- aspect that corresponds to a large number of patients with a similar level of T-cells

FIGURE 2 Focus of Jamie's utterance [56].

focus requires a well-developed intended focus, whereas the latter cannot count as well-developed as long as it is not operationalized in the form of an effective attending procedure. If so, how can the children be expected ever to build such a procedure?[9]

They certainly cannot construct an appropriate scanning trajectory in just one leap. They have to propose a procedure, test it for appropriateness and for being up to mathematical standards of adequacy, and then, if necessary, start all over again. In each such cycle, both the attending procedure and the intended focus undergo a gradual refinement. And how do the tentative attending procedures come into being in the first place? These trial constructions can be spurred only by old discursive habits, brought to mind by initial intended foci. In the new context, these former habits work as a source of hunches and intimations. It is thus the delicate dialectic of intimations and implementations that leads out of the vicious circle of focus building and allows for a discursive process in which two vital ingredients—the intended and the attended focus—co-constitute each other.

The rest of this article is devoted to analysis of the *Batteries* and the *AIDS* episodes conducted with an eye to the focus-engendering delicate dialectic of intimations and implementations and to the ways in which students learn to operate this intricate discursive mechanism.

INTIMATIONS: WHAT ARE THEY AND WHERE DO THEY COME FROM?

What Are Intimations and Why Do We Need Them?

In a simple language, intimations I deal with in my analysis are ideas for attended foci that cannot yet be fully justified in a rational way. In other words, the intimations are more or less informed guesses about the manner in which a given data display could usefully be scanned in a particular context. Thus, intimations are prods for momentary decisions, but they do not determine these decisions in an unequivocal way. The ideas that arise as a result of intimation are tentative and are intuitively acceptable but will have yet to be carefully tested and then justified in a rational way.

At this point, it seems natural to ask why we need intimations at all, and, in particular, why we need them in mathematics, seemingly the most rational of sciences. Let us take a close look at the preference problems as an example. When the children encountered this type of task for the first time, their existing discursive

[9]We seem doomed to moving in circles. The present quandary immediately brings to mind the other circles known from literature: the hermeneutic circle (see, e.g., Bauman, 1978) and the learning paradox (see, e.g., Bereiter, 1985). The dilemma here is, in fact, equivalent to the circle of reification, as presented in Sfard (1991; see also Sfard, 1998).

skills proved insufficient. For all the long practice in mathematical discourse behind them, they did not know yet how to look at the new symbolic display, such as the batteries bar diagram or AIDS dot plot, to solve the problem they faced. None of the students' former experiences seemed to fit the present situation. Whereas they had been busy, so far, investigating and juxtaposing single objects, such as numbers, shapes, and functions, they now were required to compare sets of objects, such as collections of batteries or groups of AIDS patients. This difference is quite crucial. The answer to the question "Which set is better?" requires more than establishing and comparing cardinalities of these sets. Although many properties of single objects can be measured and compared easily, sets often seem inherently incomparable because of their inner versatility and heterogeneity. On the face of it, there is only one way out of this complex situation: The students would have to follow in the footsteps of more experienced interlocutors who already know how to tackle this kind of situation. In our two episodes, however, where the teachers deliberately withheld any "showing" or "telling," there were no previously blazed, ready-made trails. Left to themselves, the students could only rely on their own intimations about possible attended foci.

The new question that now has to be asked is where the salutatory intimations can possibly come from. Let us consider the paths that one can take while tackling preference problems. If we look closely at the different solutions for the batteries and AIDS problems proposed by the children, we realize that the same general idea underlies them all: In all these cases, the children tried to translate the diverse features of the set's elements—the varying life spans of the batteries, the unequal levels of T-cells in different patients—into a measurable feature of the set as a whole. This effort is quite obvious in the *Batteries* episode, where the students explicitly declared that they were looking for an object they called "a more consistent battery," an object that was not identical with any particular battery in any of the two sets but that reified these batteries' diverse features and mutual relations. In the *AIDS* episode, the students' different inscriptions (see Figures 9–12 in the Appendix) are the translations of the original data display into one that encapsulates the dispersed sets of points into comparable wholes in which irrelevant, possibly distracting, details were omitted. In both cases, we are witnessing an attempt to construct a new object with which the entire set could be replaced. This can be seen as an effort to reduce the new situation—sets that have to be compared—to an old and familiar one—comparing single objects. If the children succeeded in their effort, the data displays with which they were working would become representations of these new objects. This is therefore, once again, an attempt to use an old experience in a new context. Naturally, because this time the old discursive habits[10] cannot readily serve the new ends, they have to undergo a certain modifi-

[10]The term *discursive habit* refers here to our propensity to react to certain situations with specific types of discursive actions.

cation before they become truly useful. Adjusting the old discursive experience to new situations is the gist of intimations.

More generally, intimations are metaphorical projections, that is, evocations of former discursive habits that are now to be applied in a situation for which they were not originally meant. They are thus the devices with which the past of the discourse adjusts itself to its present. Such intimations lie at the very heart of mathematical creativity. Clearly, they come from intended foci which, just to remind, have been defined as clusters of past experiences and discursive potentials associated with given pronounced foci. At the face of it, the best way to gain access to the mechanisms that induce intimations and then harness them to work would be to investigate the work of professional mathematicians. And yet, the mechanisms in question are extremely difficult to see and understand, not in the least because of the mathematicians' well-known tendency to present their results in such a way that the "scaffoldings" become "invisible."[11] Undoing these "camouflaging" processes is a worthwhile endeavor, one that stands a good chance of bringing fresh insights into the notoriously unyielding quandary of mathematical invention. The two classroom scenes that serve as our data provide a rare opportunity to do exactly this, even if their protagonists are schoolchildren and not research mathematicians.

Let me count the elements that have to be considered when we try to uncover the mechanisms of intimations. Basically, intimation is an association of the present situation with an experience of the past that enables a new discursive decision. The two elements that the association brings together—the present and the past—here are called *target* and *source*, respectively. Obviously, there must also be something that induces such association. The four basic elements of intimation are presented in Figure 3.

Intimations may be induced by both object–level and metalevel elements of the discourse. Usually, several factors would be responsible for a given association. Even so, in most cases it would be possible to point to one that is more influential than the rest. I now present examples of both these types, beginning with the former.

Object–level intimations are those induced in one way or another by content-related elements of the conversation. These intimations thus may be triggered, for example, by verbal clues or by visual hints found by the interlocutors in the artifacts with which they work.[12] Which element will turn into such a trigger depends, of course, on the intended focus with which a person enters the situation. The in-

[11]This description usually is ascribed to Carl Friedrich Gauss.

[12]The visual intimation is one where something we see influences the way we go on with the discourse, that is, the way we think. At this occasion it is important to stress that sometimes we are able to see an aspect of situation only because of the way in which we already think of this situation. And yet, building attended focus is an attempt to educate ourselves to see certain elements of the environment, that is, to modify our ability to notice or disregard different aspects of this environment. Thus, the activity of noticing aspects is as inherently circular as is the whole process of building the focus.

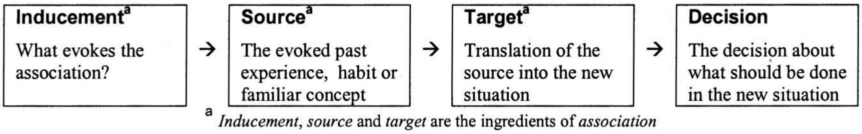

FIGURE 3 The structure of intimation.

tended focus gives salience to certain aspects of the environment while subduing some others. For example, if we are scanning the batteries bar diagram for the quality of batteries, we are likely to attend to the length of the bars but not to their width. It is also worth mentioning that several different types of inducement often would come together in a single decision. Thus, for example, a visual association may bring to mind a certain concept, and this concept, in turn, would trigger a new conceptual association. Let me illustrate all this with two detailed examples of intimations originating primarily in object–level considerations.

Example 1: Object–level intimations behind Ceasara's proposal (Batteries episode). Let us look at Ceasara's utterances with which the *Batteries* episode began. Figure 4 presents the tripartite focus that seemed to be emerging from what the girl said (see especially [8] and [12], which contain clear indications of the pronounced and attended focus).

While building her attended focus, Ceasara evidently had made three independent decisions: She chose to resort to the "range tool" (the pair of blue vertical lines with which one can establish a data range of her choice), she drew the upper limit line through the tip of the longest bar, and she placed the lower limit line in such a way that exactly half of all the bars fell within the thus created range. Taking these decisions as points of departure, I now try to unpack, one after another, the association processes from which they possibly emerged.

Pronounced focus	Attended focus	Intended focus[a]
"More consistent" batteries	1. Draw the range that includes exactly half of the bars, with the upper limit going through the tip of the longest bar. 2. Inspect one set of data (e.g. of green bars) for the number of bars within the range. 3. Repeat the procedure for the second set of data. 4. Choose (focus on) the set for which the number of bars within the range is greater and declare this set the "more consistent".	Consistently long-lasting battery = The set of batteries that has the larger subset of "longer lasting" batteries.

[a]More precisely, it is this interpreter's pronounced focus that fits the intended focus evoked in her by the speaker's utterance.

FIGURE 4 Ceasara's tripartite focus (based mainly on utterances [8, 12, 20, 22]).

Let me begin with the issue of the range tool. I view Ceasara's preference for this option as closely related to her choice of the word *consistent*, used to describe the focal object. In both cases, the underlying idea is probably the same: While facing the need to decide between two sets of batteries, one should ask not only how long are the life spans of these batteries but also how variable they are. Indeed, the thought on variability may well explain why the girl chose the range tool rather than the representative value line. It may also account for the fact that the word *consistent* appears in the discourse on batteries, in which it initially sounds rather unusual, if not out of place. Not having full historical evidence at hand, I can only speculate about when, and from where, this term got into the conversation.[13] Such speculation, even if unverifiable, may still be of value, because it sheds light on past and future possibilities. In Ceasara's case, it is not unreasonable to assume that the notion of consistency has been transferred into the pr7esent context from the discourse on human behavior. In the everyday parlance, a consistent person, or a person who behaves in a consistent way, is one who does not change her mind from one situation to another and whose actions and reactions are thus predictable and can be relied on in a longer run. This is also what one expects from a high-quality battery. This particular metaphorical projection, one that might have well been induced by a talk about the *behavior* of the batteries (yet another anthropomorphic metaphor!), is therefore not so surprising in this context, after all. The structure of this intimation is summarized in Figure 5.

The next question is why the upper limit line was located at the end of the longest bar. The answer seems immediate: The children were looking for a better, higher quality battery. The latter terms were not well defined yet, but the overpowering intuition is that in this context, *better* meant *longer lasting*. Putting the upper limit line at the right-most end of the data domain meant including in the range all the longest lasting batteries. At this point, it is noteworthy that this relatively straightforward, unsophisticated move on the part of Ceasara disclosed her inability to perform a reasonable tradeoff between the two most relevant concerns about the properties of the batteries: the concern about the consistency of the batteries' behavior and the concern about their longevity. The fact that Ceasara never questioned the placement of the upper limit line shows that despite her declared interest in the consistency of the batteries, she was, in fact, preoccupied mainly with the lengths of life spans. This, by the way,

[13]These are the shortcomings of working with somebody else's data without having access to all the records. Only after I wrote this article did I learn from Kay McClain's article technical details about the first appearance of the word *consistent* in the discourse of this class. It happened a day or two earlier, in the context of a problem which, although different, was nevertheless concerned with the issue of variability exactly like the batteries problem. This fact does not undermine my ensuing speculations on the metaphorical origins of the idea of a consistent battery.

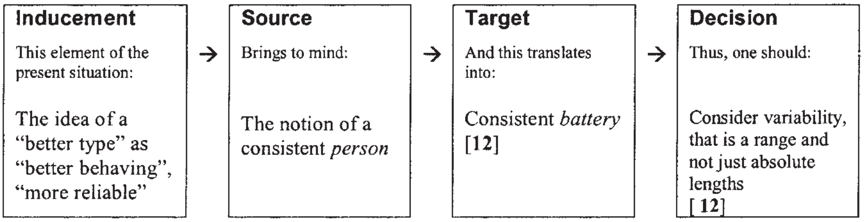

FIGURE 5 The structure of Ceasara's intimation regarding the need to consider the range.

may as well be true of other children in the class, because all of them seemed to take the upper limit of the range for granted (this is obvious to the class to such extent that some of the students mistook the range tool for the "representative value" tool, one that consists of a single line and thus splits the domain into two subdomains—see Blake's utterance [24] and Kay's [33]). For the summary of the intimation that led to the choice of the upper limit, see Figure 6.

I now turn to Ceasara's construction of the lower limit. Because this choice eventually was questioned by the class and the teacher [18, 19, 21], and because Ceasara brought some justification in response [20, 22], we have a basis for making informed guesses about the intimation that led to her decision to limit the range to exactly 10 of the longest bars. Ceasara kept in mind that the task was to compare the two sets of data. There is one clear-cut case with which she could deal easily on the basis of her discursive competencies. It is the case in which all, or nearly all, the elements of one set would be different in a well-defined way from all or nearly all the elements of the other set. Ceasara's mindfulness of there being 10 batteries of each type [20] makes it reasonable to assume that her idea "to go with the half" [22] was dictated by the wish to check the display for the possibility of nearly all bars of one color being "the longest," that is, longer than nearly all the bars of the other color. This is also what the girl declared to have found when she said that "7 out of 10 of the greens were the longest" [12]. For the summary of this intimation, see Figure 7.

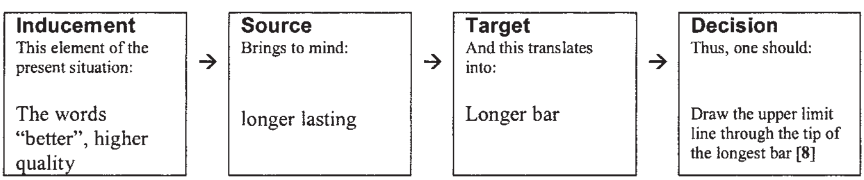

FIGURE 6 The structure of Ceasara's intimation about the upper limit of the range.

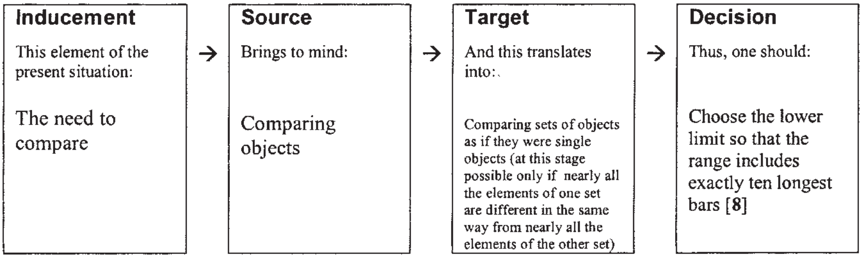

FIGURE 7 The structure of Ceasara's intimation about the lower limit of the range.

Example 2: Object–level intimations behind the solutions to the AIDS problems (the AIDS episode). The *AIDS* problem was discussed after the children prepared inscriptions that documented the way in which they proposed to scan the dot plots. These inscriptions may count as reifications of dynamic attended foci. Although we do not have direct access to the original processes of building these foci, we may try to unpack them, at least partially, from the inscriptions and from what the children said while trying to assess each of the proposals. As I show subsequently, a combination of these two sources allows the observer to make well-grounded conjectures on the intimations that gave rise to the problem solvers' decisions. Because the five inscriptions that are being discussed in this episode have been ordered in such a way that each of them, except for the first, looks like a refinement of the former one, they may be regarded as representing different stages in the process of focus building. Looking at the written records of students' thinking and listening to the classroom conversation, I identified three major intimations that seem to correspond roughly to three different stages in this process.

The first of these intimations, presented in Figure 8, is reconstructed mainly on the basis of Inscription 1 and some of the things said by the interlocutors in *AIDS* Subepisodes 1 and 2. The idea to distinguish between the two samples by looking at

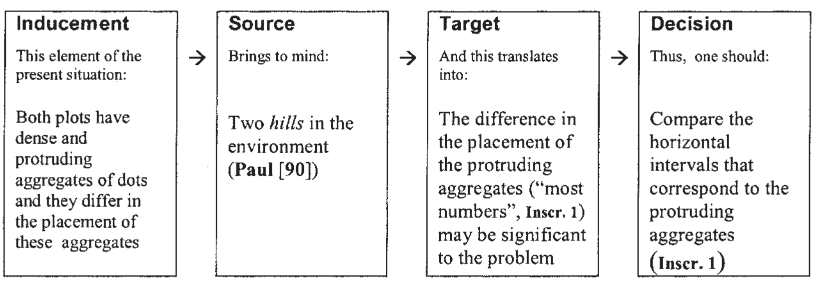

FIGURE 8 Intimation that led to Inscription 1.

the placement of "most of the numbers" might have been induced by the salient, easily noticeable visual difference between the two dot plots. Although both these plots have distinct, dense, and protruding aggregates of dots, they differ in the placement of these aggregates. This feature of the plots invokes the word *hill* (see, e.g., Paul's statement [90]), which reifies the scattered aggregates. The two sets, being considered as hill-like single objects, now may be compared according to their measurable features. The placement and extent of the hills certainly belong to this latter category. The graphic record of the first solution stresses this issue, while explicitly displaying the intervals corresponding to the "basis" of each hill.

The same salient feature of the graphic display that led to the idea of the hill evoked the familiar concept of *majority*. The majority-related intimation is summarized in Figure 9. The initial use of the terms *most* [59] and *majority* [52, 56] had the vagueness of a local ad hoc solution that does not look like a result of a rigorous reasoning or as a firm basis for a future action in a similar situation. Both terms were known to the children from other discourses, but they did not translate readily into the AIDS conversation. The students had a distinct difficulty trying to talk about them in terms of this situation [57–74], and the intimations themselves have yet a long way to go before they produce a mathematically acceptable attended focus. Because the concept of majority implies the existence of two distinct subsets, one of them bigger than the other, to operationalize this intimation one has to define the criterion according to which the split into subsets will be performed and then check which of the subsets contains more elements. The particularities of the given dot plots helped the children establish the necessary criterion. After the protruding clusters of points gave rise to the talk about hills, the interlocutors noticed a "place sort of between the hills", where one "hill started" and the other "finished" [90]. The creators of Inscription 2 chose $x = 550$ as such a demarcation point

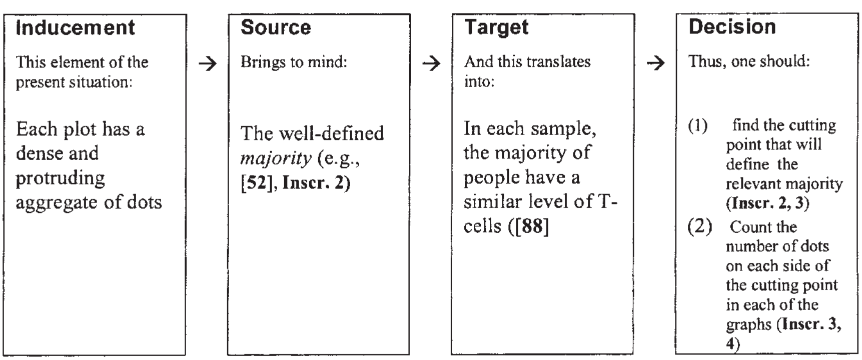

FIGURE 9 Intimation that contributed to Inscriptions 1 through 4.

[88], whereas the authors of Inscription 3 preferred $x = 525$. The latter group also counted the dots within thus created subsets, as did the producers of Inscription 4. At this point it is important to stress that this intimation and the discursive decisions it entailed are case-specific, because they make use of the particularities of the given data display. The attended focus that results would not be readily applicable in a display devoid of easily noticeable hills or in one with nondisjointed, overlapping hills. See the summary in Figure 9.

The further refinement of the notion of majority and of resulting attended focus was attained thanks to yet another metaphorical projection (see Figure 10). Jamie used the expression "like a scale drawing" [96], probably induced by an association with other situations in which pictures were scanned for their shapes. This expression implies the way of looking that disregards the actual size of the sample and attends only to its distribution. "It's not the same size, but it's the same thing", said Jamie [96]. It is interesting to see how, in later utterances [101–103, 114–116, 120], the children elaborated on the related idea of sameness-through-proportion, painstakingly paving their way from the intimation to the attended focus implied in Inscription 5, in which the absolute numbers disappear.

Metalevel intimations are ideas for discursive decisions induced by interlocutors' tendency to behave in a regular rather than accidental way that is in accord with metadiscursive rules that seem to regulate discourses.[14] Careful analysis may show a very strong influence of people's assumptions about discursive mechanisms. At a closer look, metalevel intimations are also vital for the process of learning.

Metalevel intimations, like metadiscursive rules themselves, are extremely sensitive to the issue of authority. Authoritative and nonauthoritative discursive contributions differ substantially in their power to induce metalevel intimations. By "authoritative" contribution I mean one that comes from a person who, on the force of an implicit understanding between interlocutors, enjoys a position of a relative authority.[15] In classroom discourse, this usually includes teachers, authors of instructional texts, and designers of instructional artifacts. Usually, the authority of all these parties is assumed in advance and rarely is questioned even in the most democratic of classrooms. In the episode at hand, Kay's and Paul's authoritative position as teachers was discursively signaled in many ways, one of them being their activity as exclusive discourse navigators (note that all the student-to-student

[14]Voigt (1985) would use the expression "patterns of experience" to describe the source of interlocutors' ability to act in this regular way.

[15]It is important to stress that the authority is not always seen in the same eye by different interlocutors. Usually, even if the teacher is willing to release the students from any constraints and is thus genuine in her attempts to suspend her authority, the student still will interpret her actions as authoritative, that is, as what "sets the tone." Indeed, authority lies in the eyes of the beholder not any less than in the deeds of the person seen as authoritarian.

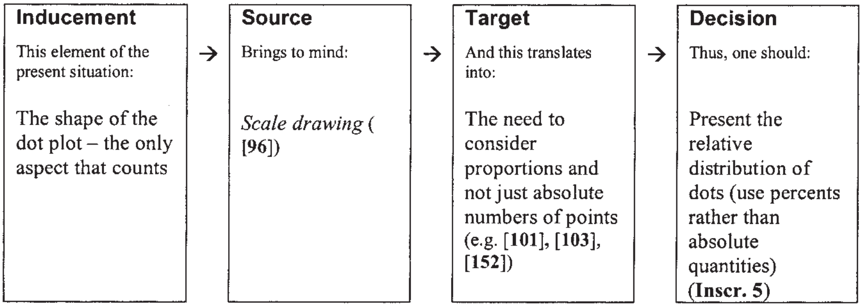

FIGURE 10 Intimation that contributed to Inscription 5.

interactions were mediated by Kay and had to have her approval; e.g., see [57–60, 122–124]).

Because authoritative contributions are a particularly powerful source of metalevel intimations, I limit myself here to this area. The first type of authoritative contribution worth mentioning is one that creates *discursive conflicts*, that is, one that signals a difference in various interlocutors' uses of words or symbols. An incongruity of uses may be hinted at, for example, by revoicing a student's contribution as a question rather than mere repetitions—a move suggesting that the revoiced contribution should be revised.[16] See, for example, Kay's interrogative "The highest range?" [68] and the student's subsequent change of mind [69]. By comparing this brief exchange to the segment [78–80] where students interrogatively revoice each other, one can see how crucial the authoritative position of the speaker is for the effectiveness of this type of revoicing. When the questioning comes from a person on an equal footing with the revoiced interlocutor, there is much less readiness on the part of the listener to interpret it as showing the need for revision.

Other types of intimation-evoking authoritative contributions are revoicings and questions that, in the eyes of the student, seem to carry metalevel hints about the speaker's preferences, even if on the object–level the latter do not seem to privilege any option. Thus, for example, in the *Batteries* episode, Kay signaled her skepticism as to Ceasara's choice of the 10 longest bars by reiterating her question several times [19, 21]. Indeed, this act was interpreted as showing enduring lack of conviction. Conversely, an idea may be rendered a special weight and become, in the eyes of the students, particularly worth pursuing, by being merely repeated by an authoritative speaker (see, e.g., [89, 90, 128, 153]). A similar type of metalevel intimation may arise from an instructional text and from various features of in-

[16]For elaboration on the notion of revoicing and its role in teachers' discourse, see O'Connor and Michaels (1996). See also the article by Ellice Forman and Ellen Ansell in this issue.

structional tasks. These intimations are the result of students' presumptions about the expected solution, stemming from their former experience with school mathematical problems. Thus, in the case of the question of preferability of a treatment, the learners are likely to expect that the one called *new* would turn out to be better. Their subsequent search of the decision procedure is likely to be influenced by this intimation.

Instructional tools are also a rich source of authoritarian metalevel intimations. These tools are reifications of the solution processes envisioned by the designers. This sounds particularly true if one looks at the different options for structuring data built into the present minitool. The users of the artifact presume they are supposed to unpack the processes from the tool. More specifically, they presuppose the possibility of an application of at least one of the existing options and are thus prepared to invest an effort in figuring out such application. This is certainly what was done by the children who proposed Inscription 5 involving partitioning the data into four equal groups. The intended unpacking, however, is not an easy task, because it cannot be guided by an understanding in any way comparable to that of the designers. Using the language introduced by Star (1989), one can say that the graphical displays are *boundary objects*, that is, products of certain discursive practice now supposed to become a basis for another discursive practice. In our case, the display is the boundary object between the discourse of researchers looking for a treatment for AIDS and of schoolchildren learning statistics. Lack of a direct access to the discourse that gave rise to the object compels the students to seek help in any available source of intimations, including those that, if made explicit, might not be considered as fully legitimate.

IMPLEMENTATIONS: WHAT ARE THEY AND WHERE DO THEY COME FROM?

Intimations are prods for discursive actions and, as such, bring about tentative attending procedures. The actual construction of an attended focus and its later critical evaluation are the components of the ensuing process of implementation.

When the *AIDS* episode began, the first part of the implementation phase—the one of constructing an attended focus—had already been completed. We can see the proposed attending procedures reified in the form of the five inscriptions. When the episode began, the teacher issued an explicit invitation to the activity of critical evaluation [50]:

> Kay: What I want to do today is I want to take a look at some of the things that you did and let's talk about them. I want to see if as a group *I want us to look at them and decide if we think that they are an adequate way to represent this*

> *data* and if we actually understand what these folks are doing. So start with one? [emphasis added]

It is noteworthy that whereas Kay framed her request for evaluation in terms of adequacy, she did it without actually explaining what may count as a criterion for this kind of assessment. It is in the spirit of her general instructional approach to leave also this decision in the students' hands. In this section, I analyze the way in which students built the criteria of adequacy for themselves. These criteria would become metarules of the new discourse that they were building. Let me precede the actual analysis with a theoretical speculation on what such criteria can possibly be. My conjectures are formulated by the way in which the students may be expected to work: The conjectures are constructed by putting myself in the shoes of the prospective user of the solution and asking myself what characteristics of the solution would make it truly helpful in solving the problem at hand.[17]

When one considers the usefulness of a solution, the first criterion for adequacy of an attended focus would be its appropriateness, that is, its being in accord with the intended focus. Howver, this cannot be all. Although the emergence of intimations and the evaluation of the appropriateness of the resulting attended focus employ associative decision making, the process of evaluation must be completed by a more rigorous testing. For a discursive construct to count as adequate, it must fulfill a number of conditions that come to ensure full consensus as to its future practical uses.[18] What criteria are likely to be applied by a battery buyer or a medical doctor (or professional statistician, for that matter) in establishing adequacy of an attended focus proposed as a solution to a preference problem? Except for the requirement of appropriateness, discussed

[17]It is highly plausible that this "practical" reasoning will lead to a set of criteria that would be acceptable by professional statisticians. On the basis of what I know about today's "canonical" mathematical discourse (cf. Burton & Morgan, 2000; Davis & Hersh, 1981; Solomon & O'Neill, 1998), I can confirm that many of the decisions actually made by the children are, indeed, grounded in criteria of this kind. This shows that a macroperspective on professional mathematical discourse sometimes may be useful in revealing mechanisms similar to those that can be found in a microanalysis of classroom mathematical discourse. This last statement should by no means be translated into the sweeping recapitulationist claim about strict parallels between ontogenesis and phylogenesis of cognition. My only assertion here is that some discursive mechanisms may be common to the professional and classroom mathematical discourse. The actual working of these mechanisms and the result they bring would, in most cases, be quite different. As I argued elsewhere (Sfard 2000b), the cycle of intimations and implementations may be the basic mechanism underlying almost any mathematical invention, either in the classroom or in the professional mathematical discourse.

[18]The concern about possible ambiguity of proposed mathematical constructs can be seen in action throughout the history of mathematics. The particularly well-documented, and also exceptionally heated, controversy between the inventors of mathematical analysis and those who, like Bishop Berkeley, complained about its shaky foundations (see, e.g., Kline, 1980) is one of the first examples that come to mind.

previously, the attended focus has to be unequivocal, generalizable, and transparent. Before elaborating on each one of these criteria let me repeat: These strict metarules for gauging the adequacy of solution had not been explicitly taught to the stats project students. Nevertheless, in both episodes we may see children spontaneously applying at least some of these rules while evaluating the proposed solutions. The rest of this section is devoted to the substantiation of this last claim. Each of the metarules for adequacy listed previously is explained and then shown in action with the help of examples from the *Batteries* and the *AIDS* episodes.

Appropriateness

For an attended focus to be deemed appropriate means to be in tune with the person's intended focus. Thus, to test a given attended focus for appropriateness, one may, for example, consider all those cases in which the solution to the preference problem seems evident and check whether the application of the proposed attending procedure would produce the same decision as the one dictated by the intuition.

In the *Batteries* episode, the children's initial intended focus grew from their intuitive idea about what counts as a high-quality battery. It found its expression in the pronounced focus on a "consistent battery." The related intended focus may be not operative enough to suggest an immediate, intuitively obvious answer to the question, "Which battery is better"? Still, it is evidently clear enough to lead to a disqualification of certain attended foci as inappropriate. Thus, when Jason reacted to the attended focus proposed by Caesara and restated by Jamie ([18]; see Figure 11) by proposing a slight change in the placement of the lower limit line—a move that overturned the former decision about the preferable kind of battery—it is because in this way he was able to show that according to commonsensical criteria, this former answer was not necessarily the most appropriate.

Pronounced focus	Attended focus	Intended focus
"Better batteries"	1. Consider all the bars that go beyond the representative-value line. 2. Find the color that dominates in this set. 3. Choose batteries of the dominant color.	Consistently long-lasting battery = The set of batteries that has the larger subset of "longer lasting" batteries.

FIGURE 11 Jamie's focus.

Unequivocalness

In mathematical discourse, an important characteristic of adequate attending procedures (as well as definitions or any other discursive construct) is that they lead to the same solution if applied to the same sets of data by different people. This is what happens if, for example, every step in the procedure is algorithmically determined. In the *Batteries* episode, this may have been the case with Ceasara's proposal, but it was not so any longer after Jason questioned Jamie's way of deciding about the placement of the lower limit line and proposed his own, nonalgorithmic method of proceeding (see summary in Figure 12; here, the word *cycle* refers to a step in the solution).

In the *AIDS* episode, the attended foci corresponding to Inscriptions 1 and 2 were certainly not unequivocal. The implicit "prescriptions" did not fully specify all the steps. They made use of what I called here "protruding and dense aggregates of points" and what the children called "hills." These prescriptions thus leave it to implementers' judgment what should count as such an aggregate and, in the case of Inscription 2, where exactly to look for the point that marks the beginning of one hill and the end of the other (in general, there is also no guarantee that a single point can play both these roles). Attended foci in Inscriptions 3, 4, and 5 are more satisfactory in this respect, because they prescribe strict algorithmic procedures for finding the range and the demarcation point.

Generalizability means that the proposed decision procedure would be applicable to any similar pair of data sets, and it would lead to an appropriate solution in every case. This feature is probably the most advanced, because it requires hypothetical considerations—considerations that go beyond the case at hand.

Attended focus (AF)	**Evaluation**
CYCLE 2: 1. Draw the range the way Cesara did 2. Inspect the bars within the range for the one that is dominant; 3. Choose the battery that dominates within the range.	Not *appropriate* – The choice of the lower limit is not necessary the best to fit the idea of consistency **(Jas, [18]**) Not *unequivocal*, since the choice of the lower limit now seems arbitrary **(Jas, [18], Kay, [19], [21])**
CYCLE 3: 1. Put the lower limit of the value range at the point $x = x_0$ in "the bunch of the close ones in the pink." 2. Consider all the bars that go beyond the line $x = x_0$. 3. Find the color that dominates in this set. 4. Choose batteries of the dominant color.	Not *unequivocal* –the choice of the lower limit is still arbitrary (can be induced from Jason's objections to Jamie's proposal, **[18]**, and from Blake's subsequent proposal for attended focus – [**19**], [**21**])

FIGURE 12 *Batteries* episode, implementation in Cycles 2 and 3—Jason's evaluation of Jamie's attended focus (Cycle 2) and proposing his own (Cycle 3).

Generalizability is also the most difficult property to test. Be the number of concrete trials as high as it can, how can a person be sure that a given procedure will produce an appropriate solution for any pair of samples? Of some help in this endeavor is the knowledge of the aspects that must be considered to ascertain generalizability. Perhaps the most important thing to know is that to be generalizable, the proposed attended focus must be independent from at least two properties of the given sample: from its absolute size and from those features of its distribution that are specific to the case at hand and may be absent from another sample. Let me attend to each of these requirements one by one.

Understanding that the results of the proposed decision procedure should be independent from the actual sizes of the given data sets is, in fact, tantamount to understanding the meaning of the word *sample*. For the students who realize that much, the given sets of data are not just concrete cases requiring concrete solutions; rather they are genuine "miniatures" of the entire population—parts that epitomize the whole. Without realizing this, the students would not have been able to cope with the AIDS problem, where the samples to be compared are of different sizes.

The issue of the size of the sample did arise in the *Batteries* episode, although it might have been omitted because of the equal sizes of the compared samples. Blake brought this aspect of generalizability to the fore while accounting for his decision to choose the batteries represented by the set, the shortest elements of which were still longer than some (two, in our case) elements of the other set. According to Blake [30, 35, 37], his choice was not between the two specific sets of 10 batteries but rather between two categories of batteries. As he made clear [37], when speaking of "the 2 or 3" short-lived batteries that will "add up" whenever there are more than 10 specimens of the kind, the boy assumed that any set of batteries of the given type would behave in the same way as the given sample, that is, would display a similar distribution of life spans. The intended generality was reflected in Blake's language that clearly related to the noun *battery* as a virtual entity and not as a tangible object. I call this virtual entity an *object* because it functions in language as one: Blake used the singular form "consistent battery" rather than "consistent batteries," even though consistency can be asserted only by inspection of a set of batteries. I call this object *virtual* because there is neither a concrete material thing nor any readily available symbolic means that may be regarded as a unique provider of its attended version. The bar display with which the children worked does not have this special status any longer: Many displays can be produced that would fit Blake's intended focus equally well. For the first time in this episode, a focus was proposed that could not be uniquely identified with any particular visually accessible thing. To put it differently, the concrete mediating tool with which the children worked became but a "representation" of "a type of battery," in itself a virtual, intangible object (as opposed to any concrete battery of this type, or to the mediating tool as such). The implementation segment of Cycle 4 is summarized in Figure 13.

Attended focus (AF)	Evaluation
CYCLE 4: 1. Look at the shortest pink bar and the shortest green bar. 2. Choose the longer of the two. 3. Choose the battery corresponding to what you found in 2 as more consistent.	*Appropriate* and *partially generalizable* in that for bigger quantities the other battery will have more and more entities that last less than 80 hours (**[30]**, **[32]**, **[35]**, **[37]**, **[40]**) The other aspect of generalizability not tested (the proposed focus is *not* generalizable to the cases with different distribution of batteries life-spans)

FIGURE 13 *Batteries* episode, implementation in Cycle 4—Blake proposing and evaluating (explaining) a new attended focus.

It is important to note that this aspect of generalizability was unique to Blake's utterances. Jessica and Sequoria, who repeated Blake's procedure and his reasoning seemingly with precision [42 and 48, respectively], did not pick up his generalizing language. For one thing, they never used the word *consistent*. Furthermore, in their explanations, the girls kept close to the concrete, immediate interpretations of the specific elements of the diagram, as if their task was to choose one of the specific two sets rather than to decide about a prospective purchase. Although both girls' narratives evolved around some clear foci, these foci were tightly related to the particular display with which the children were working at the moment.

In the *AIDS* episode, the students were forced to give thought to the issue of cardinality because of the fact that the samples they were supposed to compare were different in size. The concern about the decision-making procedure's independence from the sample magnitude was given an explicit expression in the debate about Inscriptions 3, 4, and 5, and this included Jamie's remark on data sets that were "not the same size" but were "the same thing" [96], and Will's and Blake's successive efforts to explain that actual size of the sample was irrelevant [114–116 and 127–137, respectively]. The awareness of the irrelevance of the actual size of the sample led, eventually, to the disappearance of the absolute numbers from the written records of Inscription 5, where they were replaced with percentages (also relevant to this subject are [92, 94, and 139]). The implementations in Solutions 4 and 5 are summarized in Figure 14.

The situation is rather dramatically different when it comes to the other criterion for generalizability—the applicability of the given attending procedure to different pairs of samples, and its potential for giving appropriate results also in these new cases. It can be shown easily that Blake's attending procedure, which was the last and the most refined one of all those proposed in *Batteries* episode, would not produce an appropriate result if applied to a case like the one presented in Figure 15.

In the *AIDS* episode, the attending procedures in Inscriptions 1 through 4 are not generalizable because they all draw heavily on the existence and the particular mu-

Attended focus (AF)	Evaluation
SOLUTION 4 1. Delineate the range by finding the most extreme points in both plots 2. Find the midpoint of the range 3. For each plot, find the number of dots in each half of the range 4. Choose the treatment in which the number to the right of the midpoint is greater than the number to the left	Advantage: *partial generality* – it is clear that the decision doesn't depend on the absolute size of the samples even though the absolute numbers are on display (([**103**], [**112**], [**116**]) (Other aspects of_ generalizability have not been tested.)
SOLUTION 5 1. Delineate the range by finding the most extreme points in both plots 2. For each plot, find the midpoint of the range 3. For each plot, divide the range into four parts, all of which contain the same number of dots 4. For each plot, assess the percentage of points to the right of the midpoint by multiplying the number of the parts in this half of the range by 25% 5. Choose the treatment for which the above percentage is larger	Advantage: *partial generality* – the independence from the size of the sample is stressed by the fact that the absolute numbers are not on display ([**127**], [**129**], [**136**]) (Other aspects of_ generalizability have not been tested.)

FIGURE 14 *AIDS* episode, implementation of Inscriptions 4 and 5.

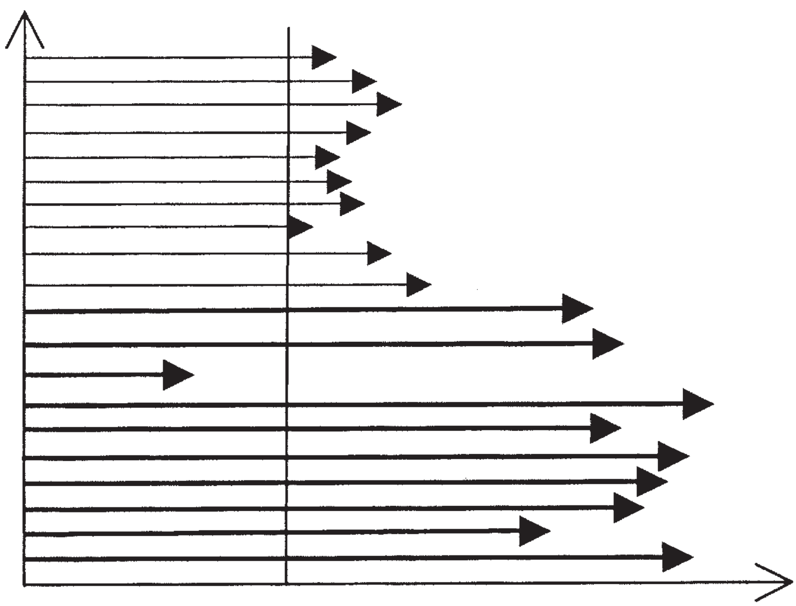

FIGURE 15 A new pair of batteries samples for which Blake's attended focus, although applicable, would not produce an appropriate choice.

tual positioning of the “hills.” In Inscriptions 1 and 2, this special feature has even been used in establishing the point of reference. All four solutions are only workable if in one sample the majority of dots concentrate in the upper half of the range and in the other sample they concentrate in the lower half. The attending procedure encapsulated in Inscription 5, although fully algorithmic and applicable to any set of data, has never been tested by the students for the general appropriateness of its results. To sum up, while looking through the restricted windows of the two brief episodes, we never see the children trying to check whether the proposed attended procedures would lead to an adequate decision also for sets of data differing in a substantial way from the example with which the class was actually working. In none of the two episodes is there any evidence that the students had constructed for themselves the requirement of generalizability, with all its entailments.

Transparency is the last feature of foci that I wish to introduce as worth considering. It is important to stress that unlike all the previous characteristics, this one is a feature of inscription rather than of the attending procedure itself. It is not definitive and is most likely to be used when one has to choose between several inscriptions supporting attending procedures that are equally adequate. Transparency means intuitive accessibility and easiness of use. There were no signs of this concern in the *Batteries* episode, but its presence was clear in the *AIDS* episode, where the children were explicitly asked about the adequacy of the solutions. The interest in the transparency can be noticed in Jamie’s utterance [52] where she stressed the possibility to see the relevant aspects of the solution; in Marissa’s utterance [92] where she claimed that Solution 3 was more “confusing” than Solution 2 because of its explicit display of absolute numbers; and in Marcus’s utterance [139] where he stressed the advantage of having the data organized into four equipotent groups.

Evaluating the proposed attended focus is an indispensable part of the solution. Developing the ability to carry out the appropriate evaluation is an important ingredient of learning mathematics. I return to this subject in the remaining parts of this article, while attempting to evaluate the learning that took place in between the two episodes.

PUTTING INTIMATIONS AND IMPLEMENTATIONS TOGETHER: HOW DO THEY WORK?

So far, I have discussed each of the various elements of the processes of problem solving separately. I now try to combine the different pieces of the puzzle and show how the intimations and implementation work together.

The intricate processes of focus building, much of which can be seen in the two episodes, can be divided into steps, all of which display basically the same structure and each of which brings a new refinement of the former solution. As such, these steps can be called *cycles*. In the focus-building process, each cycle begins with an intimation that leads to a new proposal for an attended focus and is completed with an implementation of this proposal, in which an intended focus plays the principal role. Part of the implementation is evaluation. If the evaluation ends in the verdict that the proposed attended focus is not yet fully satisfactory, it is likely to induce a new intimation and thus to give rise to a new cycle. The resulting cyclic sequence may be schematically presented as follows:

> Association$_1$ $\rightarrow$ decision$_1$ $\rightarrow$ attended focus$_1$ $\rightarrow$ evaluation$_1$ $\rightarrow$ association$_2$ $\rightarrow$ decision$_2$ $\rightarrow$

As shown later, the *Batteries* episode is a good example of such cyclic process. The *AIDS* episode, which began after the children had completed the process of focus building, cannot be used for investigating the succession of cycles in all its detail, but it does enable a partial reconstruction. I now summarize each of the two episodes so as to expose their multistep structure.

Batteries Episode

The whole episode is summarized in Figure 16, where the three elements of intimation—the inducement, source, and target—have been collapsed into the single element, called *association* (see the first column). As shown, each cycle modifies the attending procedure proposed in the preceding cycle. Naturally, one also can imagine a situation where rather than try to attempt an improvement, the interlocutors chose to give up the proposed solution and start all over again. Such a dramatic development, however, never took place in the *Batteries* episode. Rather, the process we witness is that of successive refinement. Once the suggested attended focus turns out unsatisfactory, there is a need for a new intimation that will help in making a decision about the proper way to modify this attended focus. In such process there is an ample space for students' collaboration, with everybody having an equal chance to contribute at any of the many stages of the process. The resulting focus is an outcome of such collective effort in the most essential way. No individual step in the process would be possible without those made earlier by other interlocutors, and, as a result, nobody in particular is entitled to claim an exclusive right to the invention.

I now complete the picture with a number of comments on the way in which the final evaluation in one cycle induced a new intimation of the next cycle. Let me begin with the transition from Cycle 2 to 3. It was Jason who deemed the attended focus presented in Cycle 2 inappropriate. It is interesting to note that what triggered this recognition was a mistake of his predecessor. Indeed, in Cycle 2, Jamie, convinced that she was merely repeating a former proposal, altered in fact the attending procedure proposed by Ceasara in Cycle 1. This unintended modification followed from misinterpreting Ceasara's *10* as referring to the number of bars falling within the

Cy-cle	Intimation		Implementation	
	Association	**Decision**	**Attended focus (AF)**	**Evaluation**
1	Better battery is the one that is more *consistent* (**Ceas**) Better means longer, so the range must include all the longest bars (**Ceas**) The situation is unequivocal if all the bars of one color are longer than all the bars of the other color, that is, if ten longest are of one color (**Ceas** [**20**],[**22**])	→ consider variability, that is a *range* and not just absolute lengths [**12**] → choose the tip of the longest bar as the choice of the upper limit [**8**] → choose the lower limit so that the range includes exactly ten longest bars [**8**]	1. Draw the range that includes exactly half of the bars, with the upper limit going through the tip of the longest bar. 2. Inspect one set of data (e.g. of green bars) for the number of bars within the range. 3. Repeat the procedure for the second set of data. 4. Choose (focus on) the set for which the number of bars within the range is greater and declare this set the "more consistent".	
2	Ceasara's word "ten" is interpreted as referring to the number of *the longest lasting batteries* (and not to the number of the batteries of the same kind, as it appears in Ceasara's explanations).	→ screen the ten longest bars for the color composition	4. Draw the range the way Ceasara did 5. Inspect the bars within the range for the one that is dominant; 6. Choose the battery that dominates within the range. (**Jamie,** [**16**])	Not *appropriate* – The choice of the lower limit is not necessary the best to fit the idea of consistency (**Jas,** [**18**]) Not *unequivocal,* since the choice of the lower limit now seems arbitrary (**Jas,** [**18**]; **Kay,** [**19**], [**21**])
3	There is no reason to choose *ten* as the number of bars that defines the lower limit of the range, thus it may be changed (**Jason**) The pink bars seem to be a better choice (**Jason**)	→ move the lower limit so that it would reverse the former result [**18**]	5. Put the lower limit of the value range at the point $x = x_0$ in "the bunch of the close ones in the pink." 6. Consider all the bars that go beyond the line $x = x_0$. 7. Find the color that dominates in this set. 8. Choose batteries of the dominant color.	Not *unequivocal* – the choice of the lower limit is still arbitrary (can be induced from Jason's objections to Jamie's proposal and from Blake's subsequent proposal for attended focus)
4	An unequivocal distinction between the two sets of bars may be attained by finding range that contains all bars of one color, and only part of the other (**Blake**)	→ choose the lower limit (number x_0) so that (1) all the bars of one color are above it, and (2) it is the highest of all the numbers that fulfill condition (1) [**26**]	4. Look at the shortest pink bar and the shortest green bar. 5. Choose the longer of the two. 6. Choose the battery corresponding to what you found in 2 as more consistent.	*Appropriate, unequivocal* and partially *generalizable* in that for bigger quantities the other battery will have more and more entities that last less than 80 hours ([**30**], [**32**], [**35**], [**37**], [**40**])

FIGURE 16 Association → decision → AT → evaluation cycles in *Batteries* episode.

range, rather than to the number of bars of any given type. The unnoticed mistake exposed the fact that the use of the number 10 as a criterion for the placement of the lower limit line was rather arbitrary and thus unjustified (see in particular the exchange, spurred by Jason's proposal, between Kay and Ceasara [19–22]) and induced an intimation that other placements may result in a more appropriate solution.

Let me now look at the subsequent transition from Cycle 3 to 4. Jason's proposal, although more in tune with the requirement of appropriateness than the previous one, had the salient shortcoming of being not unequivocal: It did not specify in an algorithmic way how the placement of the lower limit line should be established. The awareness of this weakness is the first factor likely to induce the new cycle in the focus-building process. Blake's intimation how the problem of the undefined lower limit line could be solved seemed to come from reasoning similar to, and yet more subtle than, that of Ceasara's. It is quite clear that both children looked for the possibility to replace the multiplicity of diverse data with considerations similar to those one undertakes when comparing single objects. And yet, although Ceasara hoped to have all (or nearly all) the bars of one set longer than all (or nearly all) the bars of the other set, Blake decided that it was enough to have all the bars of one set longer than some of the bars of the other set (as I have already noted, this request was not strong enough to ensure appropriate solution for all the possible cases).

For the sake of completeness, my evaluation of the different attending procedures proposed by the children has been summarized in Figure 17. In addition, the figure presents the numbers of utterances that evidence students' own explicit concern about the different requirements.

The *AIDS* Episode

The *AIDS* episode, unlike the other one, is the story of critical evaluation of the results of focus building rather than of the focus-building process itself. And yet, evaluation, being to a large extent an attempt at reconstructing the problem solver's

Attended focus	Cycle 1		Cycle 2		Cycle 3		Cycle 4	
	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern
Appropriate	no	[12]	no	[16]	?	[18]	yes	[35], [37], [40]
Unequivocal	yes		?		no		yes	
Generalizable	no		no		no		partially	
Transparent								

FIGURE 17 Solution quality evaluation in *Batteries* episode.

thinking, involves intimations and implementations similar to those that can be seen when the focus is being constructed. The different ingredients, as they have been identified through analysis of the classroom discussion and by looking on the written records of the solutions, are summarized in Figure 18. Although the subsequent steps, as described in the rows of the table, are not interconnected in a way

Solution	Intimation		Implementation	
	Associat.	**Decision**	**Building (attended focus)**	**Evaluation**
1	The protruding parts of the plot are the majority indicative of the quality of the treatment	Compare the placement of horizontal intervals that correspond to the protruding aggregates **(Inscr. 1)**	1. For both plots, find the interval on x-axis corresponding to the protruding and dense aggregation of points 2. Choose the treatment for which this aggregate is further to the right	Advantage: *Transparent* ([**52**]) Disadvantage: *Non-equivocal* ([**57**])
2	There is a need for a single criterion defining the relevant majority for both plots	Choose the treatment that corresponds to the hill that is more to the right	1. Find the demarcation pint between the "hills" 2. Choose the treatment corresponding to the plot with the hill to the right of the demarcation point	Disadvantage: *Non-transparent* because it is not clear why the point x=550 has been chosen ([**78**]). Possibly *non-equivocal*
3	There is a need for a single criterion defining the relevant majority for both plots	(3) find the cutting point that will define the relevant majority **(Inscr. 2, 3)** (4) Count the number of dots on each side of the cutting point in each of the graphs **(Inscr. 3, 4)**	1. Find the demarcation point between the "hills" 2. For each plot, find the number of dots in each part of the range 3. Choose the treatment in which the number to the right of the demarcation point is greater than the number to the left	Disadvantage: *Non-transparent* because uses absolute numbers ([**94**])
4	There is a need for a single criterion defining the relevant majority for both plots	(5) find the cutting point that will define the relevant majority **(Inscr. 2, 3)** (6) Count the number of dots on each side of the cutting point in each of the graphs **(Inscr. 3, 4)**	5. Delineate the range by finding the most extreme points in both plots 6. Find the midpoint of the range 7. For each plot, find the number of dots in each half of the range 8. Choose the treatment in which the number to the right of the midpoint is greater than the number to the left	Advantage: *partial generality* – it is clear that the decision doesn't depend on the absolute size of the samples even though the absolute numbers are on display ([**103**], [**112**], [**116**])
5	The need to consider proportions and not just absolute numbers of points (e.g. [**101**], [**103**], [**152**])	Present the relative distribution of dots (use percents rather than absolute quantities) **(Inscr. 5)**	6. Delineate the range by finding the most extreme points in both plots 7. For each plot, find the midpoint of the range 8. For each plot, divide the range into four parts, all of which contain the same number of dots 9. For each plot, assess the percentage of points to the right of the midpoint by multiplying the number of the parts in this half of the range by 25% 10. Choose the treatment for which the above percentage is larger	Advantage: *partial generality* – the independence from the size of the sample is stressed by the fact that the absolute numbers are not on display ([**127**], [**129**], [**135**])

FIGURE 18 Association → Decision → Building → Evaluation events in the *AIDS* episode.

that would justify calling them cycles, they do disclose internal structure identical to that of a cycle.

The *AIDS* episode is particularly useful in showing the mechanism of evaluation, as spontaneously applied by the children. My evaluation of the effectiveness of this process, as well as indications of the students' own awareness of the different requirements that must be tested, is summarized in Figure 19.

SUMMARY AND DISCUSSION: WHAT DO WE LEARN ABOUT LEARNING AND ABOUT TOOL USE FROM THIS ANALYSIS?

Along these pages, we have been thinking about human thinking as a special case of communication, and we conceptualized learning as constructing new kinds of discourse. This framework made it salient that learning mathematics includes, as one of its most substantial ingredients, the development of special discursive uses of symbolic tools. To help solve a given type of problem, the symbolic display has to be attended to in certain well-defined way. Once the proper attended focus becomes available, the symbolic artifact stops being an externally given entity and becomes an integral part of the discourse.

Developing ways of looking at symbolic displays was certainly the principal objective of the activities we observed in the *Batteries* and *AIDS* episodes. The distinct feature of the instructional approach demonstrated in these two cases was that the children were left to themselves in this endeavor, with their teachers deliberately withdrawing any explicit intervention. This gave us a glimpse into the process of mathematical creation.

The three questions asked at the beginning of this article must now be answered. Before I do this, however, I must recast the queries in the language of the communicational approach. Within this special conceptual framework, the focus

Attended focus is:	Solution 1		Solution 2		Solution 3		Solution 4		Solution 5	
	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern	As understood from graph	Student's explicit concern
Appropriate	yes		yes		yes		yes		yes	
Unequivocal	no		yes	[33]	yes		yes		yes	
Generalizable	no		no	[51]	no		partially	[69], [71]	partially	[107]
Transparent		[6]				[47], [49]				[94]

FIGURE 19 Solution quality evaluation in the *AIDS* episode.

of the investigation is on the discourse produced by the participants rather than on what "goes into their heads." Thus, the three questions asked in the beginning now have to be presented as questions about the evolving discourse. It is important to stress that these queries aim at eliciting the interlocutors' actions that modify the discourse. Let me now restate the three initial questions and try to answer them one by one.

> *Question 1*: What are the main characteristics of the adequate ways of attending to data displays such as those in the Batteries and AIDS problems?

The question may be rephrased in the following way:

> *Question 1*: What are the relevant characteristics of discourse within which one can solve preference problems? More specifically, how do graphic data displays mediate this discourse?

To answer this question one has to specify the metadiscursive rules that regulate the use of the mediating artifacts. This had to be done before I could complete the analysis of the empirical data. This is also the question the participants of the stats project had to answer themselves, if only implicitly, before formulating their solutions to the preference problems. Metadiscursive rules of the kind required in this case can be constructed only on the basis of task analysis. While considering the needs of the prospective user of the proposed attended focus, one can conclude, as statisticians usually do, that to be truly useful in preference problems, attended foci have to dictate well-defined, unequivocal attending procedures; they have to be general enough to be applicable not just to the case at hand but also to any other sample presented in a similar way; and in every one of these cases they should lead to clear-cut decisions, in tune with our sense of a "better choice" (this last requirement was called the criterion of appropriateness). In addition, it is preferable that the accompanying inscriptions are transparent, that is, intuitively accessible and easy to use.

> *Question 2*: How do problem solvers actually construct adequate ways of attending to data displays?

Or, in the communicational language:

> *Question 2*: What discursive mechanisms underlie construction of the discourse that leads to the solution of preference problems? More specifically, how do data displays come to play their role as discourse mediators?

Our analysis revealed that the mechanism underlying creation of adequate attended foci involves an intricate interplay of intimations and implementations. While eliciting many different kinds of intimations, this analysis has made abundantly clear the central role of the former discursive experience of any kind, not just mathematical.[19] The object–level associations are not the only possible type of intimations, though. In our investigation we were also looking for the hidden ways in which discursive contributions of teachers and tool designers can influence the student's decisions, sometimes against their intentions. This kind of discursive impact was found in both episodes. Mathematically minded persons may be alarmed by the use of associations and metalevel clues rather than rational considerations. Moreover, reliance on metadiscursive hints coming from particular texts and artifacts increases the situatedness of learning. On the other hand, there is no point in trying to avoid decisions based on the metaphors and metalevel intimations, because these are not only inevitable but also irreplaceable in overcoming the inherent circularity of the task of focus building.

In the cyclic focus-building process, the evaluation of the proposed attended foci was an important part of the discursive activity of implementation. It is here where the new metadiscursive rules mentioned in the response to Question 1 had to be applied. Evaluation of the proposed attended foci according to the criteria of appropriateness, unequivocalness, generalizability, and transparency is what the stats project participants evidently did in both episodes, even if it was done in a somewhat partial and mostly implicit way.

> *Question 3*: How much learning occurred between the *Batteries* and the *AIDS* episodes? More specifically, how skillful did the stats project children become in constructing adequate ways of attending to data displays?

Provided the term *learning* is interpreted as becoming a skillful participant of a new discourse, this last question does not seem to require reformulation. Let me stress once again that learning expresses itself not just in the actual extension of the discourse but also in the mastery of the mechanisms that underlie such extension and may lead to constructions of the same kind in the future.

Let me begin my assessment of learning with a disclaimer: An attempt to evaluate learning on the sole basis of the two very brief and restricted glimpses into the ongoing discourse of the class has obvious shortcomings. Some elements of learning that are crucial for evaluation might have been invisible to us not because of their absence but because of the limitations of our field of vision. For the sake of

[19]The value of associative thinking is invariably stressed by research mathematicians, whenever they are asked to explain the mechanism of mathematical invention. To give just one example, let me bring a quote from my own conversation with a mathematician (originally reported in Sfard, 1994): "When you ask me whether something is true or not, I can think about it a moment…*find a similarity to something else*…and I can give you an answer out of the sleeve. And all this when I have no inkling about a proof [emphasis added]."

the evaluation of learning, our present data seem insufficient in a number of ways. First, neither of the two episodes gives us an opportunity to see the class arriving at a final consensus as to which solutions should be preferred and why. This does not mean that such consensus was been attained. It only means that our data are too restricted to let us know for sure. Another related shortcoming of the available empirical material is that neither of the two brief scenes shows the completion of the process of focus building, because none of the proposed attending procedures is fully satisfactory from the prospective user's point of view (i.e., does not meet all the criteria listed in the answer to Question 1). Once again, this is likely to be a weakness of the picture rather than of what is being pictured. All these restrictions and limitations notwithstanding, I attempt a tentative evaluation of the learning that took place to demonstrate the principles of the kind of assessment that is made possible by the type of analysis presented in this article.

Let me first try to evaluate the learning of the class as a whole. Our findings indicate the collective ability to put the cyclic process of intimations and implementation in motion. This latter ability can be decomposed into two complementary discourse-building skills, the associative and the deductive, both of which can be closely observed in the *Batteries* and *AIDS* episodes. Whereas associative discourse building—the one that demonstrates itself whenever a new intimation enters the scene—is a rather unlikely object of systematic instruction, the rational decision making can be deliberately fostered by making explicit the rules that underlie the focus-evaluating procedures. By comparing summaries of students' evaluating activities displayed in Figures 16 and 18, one can see that from the *Batteries* to the *AIDS* episodes, the class as a whole made much progress in applying the criteria for the adequacy of solution. The proposed attended foci improved in quality, and in the *AIDS* episode the students seemed more aware of the need for appropriateness, unequivocalness, generalizability, and transparency of the solution (although none of these requirements was explicitly named) than they did in *Batteries* episode. And yet, students' ideas about generalizability seem still incomplete, because in neither of the two episodes do we witness any attempt at examining the applicability of the proposed solution to another pair of samples. To sum up, although the students, as a group, showed a reasonably well developed awareness of what counts as mathematically adequate decision procedure, there is room for further improvement.

The next natural issue to raise is that of individual learning. More specifically, the question is whether the collective problem solving improves all the necessary discursive skills in individual children. Also of interest is the issue of individual forms of participation. Some students might have shown particular strength in only certain parts of the process, and thus different students might have contributed differently to the collective focus building. Unfortunately, the available data are much too restricted to support or to refute this conjecture. Only three children–Blake, Jason, and Jamie–appear in both episodes, whereas Jason and Jamie are not sufficiently "visible" to justify a reasonable evaluation. One student, however, Blake, can be seen in both episodes making important, insightful, and rather eloquent contributions. According to this analysis, Blake seemed to have come nearer than anybody else in the class to the kind of display use that can be regarded as adequate. He seemed to have mastered the intricate mechanisms of focus building. The only aspect that needs further care is Blake's awareness of the need for fully-fledged generalizability. As far as we could see, he did not show any concern about the applicability of the proposed solution to cases other than the one at hand.

For the sake of an exercise in on-the-run instruction planning, let us assume that the preceding assessment of learning is just and that no additional learning has yet taken place. In this case, our observations so far give us a good idea about the ways in which the students could now be helped in their attempts to develop their discursive abilities even further. Thus, for example, to strengthen and develop students' awareness of the need for unequivocalness and generalizability of the decision procedure, one could present them with a number of additional examples, with quite different distributions of data. To minimize situatedness and promote further development of the "canonical" use of symbolic tools, other types of data-rendering devices should be introduced. If these other devices support qualitatively different attended foci, they will neutralize the impact of the particularities of specific displays. Finally, those responsible for the course of actual classroom conversation might use their new alertness to the situational aspects likely to influence students' discursive decisions.

CONCLUDING REMARK: ON THE INSEPARABILITY OF THOUGHT AND TOOL

Perhaps the most important conclusion from the approach to cognition and from the subsequent analyses offered in this article is that artifacts we use as discourse mediators are much more than "aids to thought"—they are thought enablers and generators and are inseparable from the thought just like our physical actions are inseparable from our own body and the tools we use. Within the communicational approach, it is thus rather senseless to make such statements as "the same thought has been conveyed by different means" (that, however, does not mean that we cannot interpret two expressions in the same way, with *interpretation* and *thought* being two different things). If thought is discourse, and if the discourse is inseparable from its mediating tools, there is no "cognitive essence" or "pure thought" that could be extracted from one symbolic embodiment and put into another. This conclusion, as philosophical as it may sound, has important practical entailments. One of them is that the nature and quality of thought is a function of the nature and quality of the mediating artifacts, just like the nature and quality of our physical action is a function of the nature and quality of the material tools we use.

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