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Visualizing the emergent structure of children's mathematical argument

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Abstract

Mathematics educators suggest that students of all ages need to participate in productive forms of mathematical argument (NCTM, 2000). Accordingly, we developed two complementary frameworks for analyzing the emergence of mathematical argumentation in one second-grade classroom. Children attempted to resolve contesting claims about the "space covered" by three different-looking rectangles of equal area measure. Our first analysis renders the topology of the semantic structure of the classroom conversation as a directed graph. The graph affords clear "at a glance" visualization of how various senses of mathematics—as imagined, as performed, and as historically rooted—were interrelated. The graph represents the emergence and intercoordination between conceptual and procedural knowledge in the ebb and flow of classroom conversation. The graph has not just descriptive, but also predictive power: Interconnectedness among the nodes of the graph representing the first 40 min of conversation predicted the structure of recall in the final ten minutes by children who had played little overt role in the conversation up to that point. The second, complementary framework draws upon Goffman's expanded repertoire of roles in speech to demonstrate how the teacher orchestrated this classroom conversation to establish coherent argument. In this second analysis, we establish how the teacher mediated between the everyday talk of her students and the discourse of mathematics. Her revoicing of student talk created interstices for identity and participation in the formation of the argument. Together, the two forms of analysis illuminate the emergence of mathematical argument at two levels: as collective structure and concurrently, as individual activity.

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1. Introduction

Although mathematics is often viewed as the discovery of pre-existing structure, an alternative perspective is that mathematics develops within a history of collective argument and inscription (Davis & Hersch, 1981; Kline, 1980; Lakatos, 1976). From this latter view, one of the goals of a mathematics education is to help students learn to participate in the production of mathematical arguments. These opportunities for learning should be available from the inception of schooling, not reserved for "later" or "advanced" study (NCTM, 2000). This is an ambitious goal for both educators and researchers. For researchers, it provokes the need to generate new analytic frameworks for the study of innovative forms of teaching and learning. Here we introduce two frameworks developed to examine the unfolding history of a mathematical argument during one lesson in a second-grade classroom. As we will describe, the history of this argument involved the emergence and intercoordination of conceptual and procedural knowledge, so its conduct also provided a forum for the study of how these forms of knowledge can be mutually supportive in mathematics education (e.g., Rittle-Johnson & Siegler, 1998; Schoenfeld, 1986; Silver, 1986).

The teacher of this second grade class participated in a program of professional development to help children construct a mathematics of space and geometry in the primary grades. In this program, effective teaching began with children's everyday experiences, which were then progressively "mathematized," accompanied by a press from the teacher for generalization and certainty (e.g., "Does it work for all cases?" "Can you know for sure?"). Procedures were grounded in meaningful distinctions among forms of activity, rather than being learned in isolation, and mathematical ideas were developed and reified through talk and inscriptions (Lehrer, Jacobson et al., 1998; Lehrer, Jacobson, Kemeny & Strom, 1999). Hence, this classroom was an excellent forum for investigating early forms of mathematical argument. Here we focus on a single lesson, in which children first explored the mathematics of area and its measure.

Our purposes are twofold. First, we intend to describe an analytic framework that reveals how different mathematical senses of area measure are developed and *structured* over the course of the lesson. The structural analysis is an analog to a cognitive task analysis, with its attendant description of a problem space (Newell & Simon, 1972). The analysis produces a visualization of the structure of the argument being made, so that we can see what children could conceivably learn by participating in it. This approach is consistent with the tradition in cognitive science of describing the mental environment or landscape to which cognitive agents adapt. As Simon (1969) points out, in education these are cognitive adaptations to designed, rather than to naturally occurring, environments. We refer to classrooms as designed environments because in these contexts, learning depends on teaching (and vice versa), rather than solely on aspects of the cognitive architecture.

The coupling of teaching and learning suggests a second goal: to characterize how the classroom teacher helped children participate in the creation of the argument whose structure is the focus of the first analysis. Of course, children can readily develop persuasive and defensible arguments, but these arguments typically rely on untested presumptions of shared knowledge that from a disciplinary perspective, may not constitute adequate grounds of evidence (Anderson, Chinn, Chang, Waggoner, & Yi, 1997). The teacher's assistance was also essential because this argument about area extended over a prolonged period of time and required the integration of many sources of information. These conditions made it difficult for children to maintain relations between their conjectures and evidence, a problem that even adults encounter when constructing sound arguments, and that children often find especially difficult (Kuhn, 1989; Schauble & Glaser, 1990). Hence, the analysis of how the teacher rendered assistance constitutes a necessary complement to the structural analysis, showing *how* the structure was generated and maintained.

We view these two levels of analysis as complementary, much as biologists might take organism or population perspectives on the "same" event (Mayr, 1997). Accordingly, in the first section, we describe the development of an analytic framework—a directed graph—to render conceptual development visible, and ultimately, to make the structure of the argument inspectable, an object for further investigation. As we shall relate, the graph displays coordination among concepts, procedures, and a previous history of learning as these emerged during the course of classroom interaction. The interactions were intended to resolve contesting claims about area and its measure. The level of analysis is collective; that is, we describe the argument as it developed across the entire class. Yet, we will also show that this collective structure predicted what a small group of individuals recalled about the argument made by the entire group. Next, we turn to an account of how the teacher orchestrated the structure revealed by the directed graph. This second analysis is focused on individuals, although these individuals act in concert. It traces how the classroom teacher "revoiced" (repeating, expanding, rephrasing, or reporting) student talk to transform student utterances into forms that were more mathematically fruitful (O'Connor & Michaels, 1993; O'Connor & Michaels, 1996).

2. The structure of the argument

In mathematics education, there is ongoing concern with how students and teachers alike appreciate various forms of mathematical argument, especially proof (Harel & Sowder, 1998). Schoenfeld (1989), for example, suggests that instead of understanding proof as a form of explanation, many students conceive of it as a means for arriving at the truth of the empirically obvious. International comparisons of students (e.g., Healy & Hoyles, 2000) confirm this point, and apparently, many teachers hold similar views (Martin & Harel, 1989). These findings indicate a need to consider the epistemological foundations and corresponding forms of mathematical argument. Although these issues are typically addressed with high school and college students, an emerging trend in mathematics education is to foster mathematical argument in the elementary grades, so that children can come to understand different forms of mathematical explanation (e.g., Ball & Bass, 2000; Lampert, 1990; Cobb, Wood, & Yackel, 1993). Collectively, the "design experiments" (Brown, 1992; Cobb, 2001) that support these kinds of reasoning emphasize that classroom talk is the "medium in which claims are developed, made, and justified" (Ball & Bass, 2000, p. 205). Yet the medium is not neutral; in these classrooms, talk is more than an instrument for transmission. Teachers must actively orchestrate "norms" governing talk—for instance, about what constitutes a mathematically valid explanation (Yackel & Cobb, 1996). Teachers *revoice* (repeating, expanding, rephrasing, or reporting) student utterances in ways that juxtapose temporally discrete claims (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; O'Connor & Michaels, 1993, O'Connor & Michaels, 1996), and they engage children in hypothetical or suppositional reasoning about mathematical ideas and objects (Ball & Bass, 2000; Lehrer et al., 1998). Common to these teaching moves is an effort to create a public base of knowledge, shared by the class, that can subsequently serve as a resource for more elaborate forms of argument. The ultimate aim is a mathematics education that cumulates across lessons and years.

Our first analytic effort is aimed at making more visible the mathematical structure that emerges from these efforts to have students justify claims and engage in hypothetical thought. The framework created for this analysis is informed by studies of the development of mathematical reasoning that emphasize the interplay between conceptual and procedural forms of knowledge (procedure here refers not to a claim about knowledge representation, but rather to knowledge of particular mathematical actions). But it is also grounded in theories of the semiotics of mathematics that emphasize an interplay among concepts, procedures, and community in the professional practice of mathematics (e.g., Rotman, 1988; Rotman, 1993).

2.1. Brief synopsis of the lesson

We analyzed a single 50-min lesson to explore the utility of representing conversation in the classroom as a directed graph representing mathematical concepts, procedures, and histories. The teacher began the lesson by inviting children to look at three paper rectangles (unlabeled unit dimensions of 1×12, 2×6, 3×4) displayed on the blackboard, and to consider which of the three might cover the most space. The problem was anchored in a previous classroom history of designing quilts, so that the rectangles were referred to as pieces of a quilt. The units of measure for the rectangles were "quilt squares' (e.g., a 1×12 piece was made by horizontal tiling of 12 squares), but as will become apparent, the unit was not initially obvious to children. When asked which piece covered the most space, children shouted, "A," "B," "No, C," referring to the labels associated with each rectangle. Initial justifications focused on which rectangle was "skinniest" or "fattest."

These contested claims were developed and resolved by various means of partitioning and rearranging the three forms. Initially, children relied on gesture and matching parts of shapes to demonstrate that the shapes were additively congruent (i.e., the parts of one rectangle could be rearranged and composed to establish congruence with another rectangle). Later, they developed *templates,* demonstrating that two or more forms could be composed by repeated iteration of the same template an identical number of times. Finally, drawing upon the class history of quilt design, students constructed a unit a measure, a *core square,* and found that all three rectangles had the same measure in units of core squares. The procedures invented by the children led to progressive refinement of their conceptions of "space covered by" a form, to the point where all children were persuaded that the implausible could be true: The three forms all had the same area measure. Forms may "look different, yet cover the same space." Reaching this conclusion was neither inevitable nor rapid. It was achieved with the assistance of the teacher, who orchestrated the discussion by foregrounding different

Table 1
Conceptual knowledge of area measure

Code	Label	Example:
C1	Space covered	"They cover up the same space."
C2	Congruence-Area Distinction	"They look very different 'cause one is skinny and one is fat."
C3	Additive congruence	Implication that if congruence can be established after making and rearranging parts, the shapes cover the same amount of space.
C4	Size and count of units directly related	"But Sam's using the long side (a bigger unit), not the short side. That's why he only gets 3."
C5	Area measure is count of units	"They're the same, because they all have 12 squares."

students' beliefs, by skillfully juxtaposing conflicting claims, and by encouraging students to examine the entailments of their views.

We analyze the chronological emergence of concepts about area and its measure in the shared discourse of students and their teacher. As noted, our analysis focuses on the development of this argument. We show how conceptions and procedures mutually bootstrap to produce a sound conclusion. We treat the classroom as a unit by accounting for the development of the argument as a whole, and consider how each utterance made by students or the teacher contributed to the progressive mathematization of area.

2.2. Characterizing mathematical functions of classroom talk and action

We distinguish among three mathematical functions of classroom talk and corresponding activity. First, we consider the senses of area familiar to mathematicians, who assign to any planar figure a number that represents the ratio between the "size" of a figure and the size of a given planar unit (usually a square). Establishing this ratio involves tiling the planar figure with an appropriate unit and counting the number of these units. (We ignore here multiplicative senses of area involving products of length. These were considered in subsequent lessons.) Thus, area measure coordinates spatial and numeric reasoning. We distinguish among five different senses of area noted in studies of children's cognitive development, beginning with informal notions like "space covered by," and culminating in the emergence of the conventional units of area measure (Lehrer, Jenkins & Osana, 1998; Outhred & Mitchelmore, 2000; Piaget & Inhelder, 1956; Piaget, Inhelder, & Szeminska, 1960). Table 1 displays the categories we developed to accommodate these different senses of the concept of area.

Second, we consider the kinds of mathematical procedure that might accompany the development of a mathematics of area measure, including spatial operations involving comparing, partitioning, rearranging, and establishing the congruence of two or more planar forms. Our coding of procedures was also informed by studies of cognitive development,

Table 2
Procedural knowledge related to area

Code	Label	Description of Action
P1	Rudimentary Comparison	Looks at appearance. “A is taller.”
P2	Compensation	Looks, suggests compensation. “A is longer and thinner, But B is shorter and fatter.”
P3	Rearranges parts	Rearranges parts to establish congruence but parts are not all congruent.
P4	Matching parts	Congruence established by part-to-part correspondence. Parts within shape are not all congruent.
P5	Traced part used as template	Tracing of part used as template for establishing corresponding parts.
P6	Form congruent partitions within shape	Folds shape into congruent partitions, like halves, thirds, fourths.
P7	Use template to form congruent partition between shapes	Use one of the congruent partitions from one shape as a template for dividing a second into congruent partitions.
P8	Count parts in partitions	“Can you get four pieces this big out of that shape?”
P9	Comparing partitions	Parts, size, or congruence of partitions are compared.
P10	Count units	Number of units quantifies area.
P11	Reconfigure templates as units	“There are four of these parts, and each one is 3 squares.”
P12	Counting groups of units	Skip-count units in groups.
P13	Unit as unit of construction	Units used to compose new shapes of equal area.

although some of the procedures were particular to this task. We identify thirteen distinct procedures marked by classroom talk, beginning with perceptual comparisons of the relative sizes of two or more shapes and concluding with the development of methods that employed congruent partitions to serve as quantifiable units of measure. These procedural distinctions are displayed in Table 2.

Third, we consider the ways in which children’s prior history grounded the activity described in this lesson. Here we include general reference to a previous history of the geometry of quilt design (e.g., “When we worked with the quilts”) as well as reference to specific episodes and notations developed in that earlier context. Explicit marking of historical reference in conversation, as summarized in Table 3, allows us to determine the extent to which these histories played a vital role in the construction of this argument, rather than being mentioned in passing. We also coded student talk about repurposing a unit of quilt design (a “core square”) to serve as a unit of measure (the final category in Table 3). This final category of talk represented a confluence of history (the core square), procedure (counting), and concept (area as quantity).

Table 3
Prior history and notation

Code	Label	Example/Description
Q1	General context	"I'm working on three quilts."
Q2	Amount of cloth	"So which takes more cloth to make?"
Q3	Core square	"It's a core square."
Q4	Core square construction	Mention that same number of core squares can be used to build quilts of different shapes.
N1	Notation	Core square is inscribed and used as a unit of measure

We parsed approximately 50 min of classroom talk into turns marked by different speakers and coordinated this parsing with its corresponding (digital) video record. The video record helped contextualize the conversation with the objects of visual regard, including accompanying gestures and other nonverbal cues. After segmenting the talk and the video, we coded each turn according to the scheme outlined in the Tables (three raters, two of whom were independent, with 84% agreement). Representing each code as a node, we developed a directed graph representation of the flow of events for the entire lesson. The directed edges of the graph indicate temporal sequence of the codes, with numbers labeling the chronological order of each edge. For ease of display, we parsed the lesson into five ten-minute segments, and a sixth concluding two-minute segment that initiated a follow-up lesson. In the next section, we describe each segment of the lesson.

2.3. Constructing a directed graph of the emerging argument

In this section, we display the intercoordination among categories of concept, procedure, and history as a directed graph. The graph models the structure of the mathematically significant portion of events that evolved over time. This structural description assumes the role of a task environment for subsequent analysis. We briefly overview each the six segments of the lesson, display each segment's directed graph, and note highlights of mathematical significance, as they are illuminated by the graph.

2.4. Segment 1: provoking contesting claims

2.4.1. Overview

Children saw three different rectangles, called quilts, pinned on the blackboard, and were asked to decide which "needed the most cloth" to make. The assignment was contextualized broadly by reference to a preceding instructional unit on quilt design. During this prior unit, children made quilts from core squares and geometric transformations of these squares, such as sliding, turning, and flipping. As they designed quilts, children had ample opportunity to explore geometrical shapes (squares, rectangles, triangles, etc.). Children's experiences with quilt design and geometric transformations provided conceptual and procedural resources that their teacher exploited during the course of the current lesson.

The first reactions of the students focused on differences in appearance, but during this time, at least one student proposed that perhaps the three rectangles all “covered the same amount of space.” This conjecture led to some rudimentary comparisons, followed by the suggestion to rearrange parts of one rectangle to see if these parts would cover a second. In spite of the apparent sophistication of this suggestion, the teacher believed that the students did not understand the implications of their actions. For this reason, she chose to highlight the concerns expressed by several students, asking children directly about the differences in the rectangles’ appearance. We turn now toward representing this flow of events as a directed graph.

2.5. Visualizing the segment

The visual display that represents this and all subsequent segments of the lesson was constructed as follows: First, each conversational turn was coded according to the scheme outlined in Tables 1–3, so that different mathematical aspects of the turn were explicitly represented. Each of the 23 categories was represented as a node on a graph, and the nodes were ordered in a circle. Fig. 1 represents the semantic progression during the first 10 min of the lesson. The bottom half of Fig. 1 represents 13 procedures (*P1-P13*) carried out or described by the participants during the entire lesson, arranged so that the mathematical sophistication of the action (from the perspective of the discipline of mathematics) increases counterclockwise along the arc. For example, rudimentary perceptual comparisons, such as comparing rectangles with one’s “eyes,” anchor the left end of the arc. Actions involving partitioning (e.g., dividing the rectangles into regions) and then matching parts to establish equivalence of area between two or more rectangles fill out the center of the arc. Procedures involving manipulation of templates (i.e., privileging one part) or using units to establish equivalence of area complete the right end of the arc.

The remaining codes are represented as nodes in the top half of the circle. The top left arc of the circle (*C1-C5*) represents different conceptual senses of area, arranged so that informal notions of “space covered” occupy the left end of the arc. Moving clockwise, nodes representing ideas about the relationship between congruence and space, especially the idea of *additive congruence*, fill the center of the arc. Additive congruence means children claim that shapes cover the same amount of space if both can be partitioned so that each of the parts in one shape has a corresponding congruent part in the other shape. In general, children of this age accept that if two shapes are congruent, then the shapes must take up the same amount of space. But they also tend to believe that the sufficient condition of congruence is a necessary one: That is, they believe that if two shapes take up the same amount of space, they must be congruent (Lehrer et al., 1998). Thus, the emergence of the notion of additive congruence is a significant conceptual development. Ideas of area as a quantity that is determined by a count of units, and that the relative size of the unit affects this count, complete the remaining portion of this arc of conceptions. The top right arc of the circle (*Q1-Q4*) represents references to the shared history of quilt design. References to this previous history are ordered so that general reference to quilts as physical or designed objects appears at the left end of the arc; reference to the amount of cloth in a quilt or to core squares

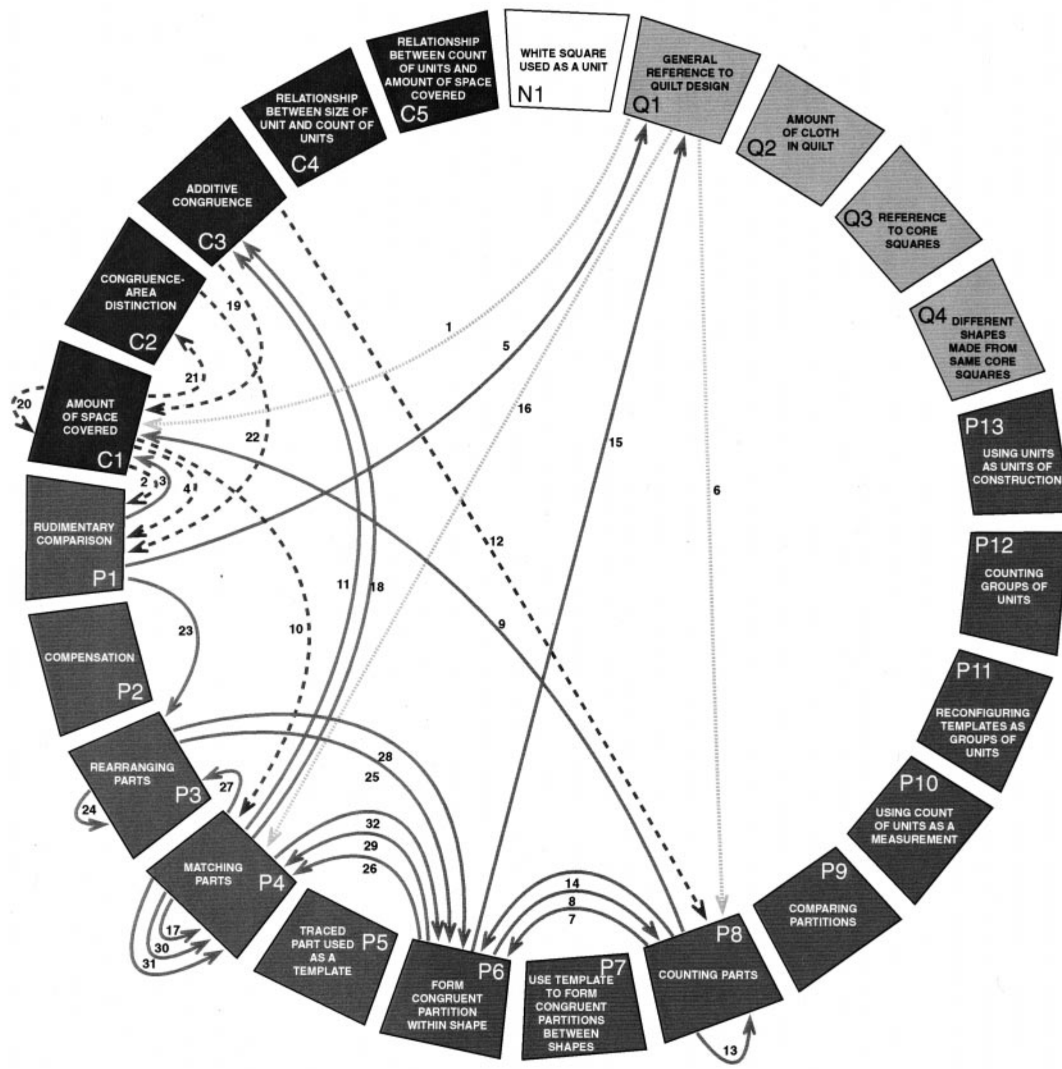


Fig. 1. Graph of segment 1 of the lesson.

(an object used to generate quilts) appears at the center of the arc; and explicit recall of designing different-shaped quilts from the same number of core squares appears at the right end of the arc. The remaining category, the white node in the top center (*N1*), refers to the use of notation (i.e., a core square). The use of notation repurposes previous history (i.e., the core squares take on a new function) and entails the concept of unit and related procedures involving iteration. Notating units increases their mobility and stability (Latour, 1990). That is, one can literally pick the units up and move them around for purposes of measuring. Moreover, the unit notation “freezes” the measure for children, whose conceptions of unit are still fragile.

2.6. Tracing the argument

The directed edges of the graph displayed in Fig. 1 indicate temporal sequencing, that is, the arrows do not necessarily represent any particular utterance, but do indicate when one code is followed by another in the transcript. Therefore, the graph represents the chronological flow of the argument across the collective of teacher and students. The shading of the edges corresponds to the shading of the originating nodes. The numbers on the edges of the graph represent the order of the connection in the coding system. For example, the first code on the transcript is *Q1,* referring to the teacher's general reference to the previous history of quilt design: "I have quilts. . . I'm working on three quilts at once. . ." Shortly thereafter, the teacher went on to say: "What I want us to talk about is how much space each of those shapes covers," which we coded as *C1*. Consequently, the first arrow, labeled with a 1, is drawn from the node *Q1* on the graph to the node *C1* on the graph (See Fig. 1). The class proposed several rudimentary means of comparing the space covered by each quilt based on height or width, so that the interplay between looking and space covered is captured by the second through fourth edges of the graph. Then a student referred to prior experiences designing quilts (the fifth edge terminates in *Q1*), and talked about the rectangles as composed of parts (the sixth edge).

The remaining edges of the graph correspond to the evolution of the argument to include other senses of equivalence. For example, the teacher asked, "How will thinking about the parts help us to know which of these shapes takes up the most space?" A student proposed actions of partitioning and then explicitly matching parts between two of the shapes. Several other students suggested that these actions implied that shape A and shape B covered the same amount of space, a notion that we call additive congruence (code *C3*).

Then the teacher asked: "How can they look so different and cover the same amount of space?" This remark made explicit a tension between the strict mathematical concept of congruence and the more general notion of same area measure (code *C2*). (Mathematically, congruence is a stricter condition, because it applies to a smaller subset of objects. Psychologically, congruence is more basic, because it is directly perceivable.) Students then attempted to use actions of partitioning and matching to show that one of the shapes was bigger than the others.

Fig. 1 shows that during this first ten minutes of the lesson, most of the procedures invented by students involved rudimentary comparison of the three rectangles, by partitioning shapes (by folding) or matching parts of the shapes. These procedures were related to the mathematical concepts of amount of space covered by the shape, additive congruence, and the potential conflict between appearance (lack of congruence) and reality (area measure). The references to history simply contextualize the rectangles as quilt pieces. No other aspects of this prior history were invoked, either by the teacher or by any student.

2.7. Segment 2: appearance-reality conflict

2.7.1. Overview

This segment began with a student explaining the apparent conflict between appearance and reality by noting that one of the shapes was short and fat and the other, tall and thin.

Several students were not convinced by this argument based on perceptual compensation. They proposed rearranging parts to demonstrate that shapes B and C were additively congruent. At this point, there was a segue of approximately five minutes in the discussion, during which students discussed properties of rectangles and squares. Although this detour was mathematically productive (children invented a means to distinguish squares from rectangles), it did not focus on ideas about area measure, and so we did not code it or include it in the analysis. Following this segue, a student proposed another means of partitioning, matching, and rearranging parts to establish additive congruence.

2.7.2. Tracing the argument

Fig. 2 shows that during this second segment of the lesson, children revisited many of the same procedures and concepts enacted or discussed in the previous ten minutes. There was no explicit mention of prior history during this segment. The graph representing the second segment is relatively sparse because we did not code the tangential discussion about properties of rectangles.

2.8. Segment 3: constructing templates to resolve appearance-reality conflict

2.8.1. Overview

The segment began with one student elaborating a procedure for comparing the space covered by two rectangles. He formed a template by tracing one part of shape B on the blackboard, and the class overlaid each of the parts (from the previous partitions of shapes B and C) on top of this template, using the template as a standard to determine the congruence of the parts. Several students then noted that this procedural sequence implied the additive congruence of this pair of rectangles "because they have the same parts."

The teacher next drew the students back to the contextual grounding of quilt design, and asked them which of these two quilt patches "takes up more space." Even though the students had just performed three different procedural sequences indicating that these two shapes were additively congruent, nevertheless, many students continued to claim that one of the quilt patches took up more space than the other, because "they look different." Responding to these competing claims about "size," a student partitioned shape B by folding it into two congruent parts, and then repeatedly laid one of these halves (the template) upon shape A to mark a congruent partition of shape A. After this sequence was completed, another student generalized this template method. He partitioned shape C into fourths and used one fourth to create a congruent partition of shape A. Then the teacher focused the class's attention on the number of parts in the resulting congruent partition.

2.8.2. Tracing the argument

Fig. 3 indicates several distinct transitions in the development of the argument. First, relationships were established between the previous history of quilt design and mathematical conceptions (C1-C3). Second, the procedure of matching parts was connected explicitly with the concept of additive congruence. Third, the class drew relationships between partitioning and template procedures (templates were refinements of partitions).

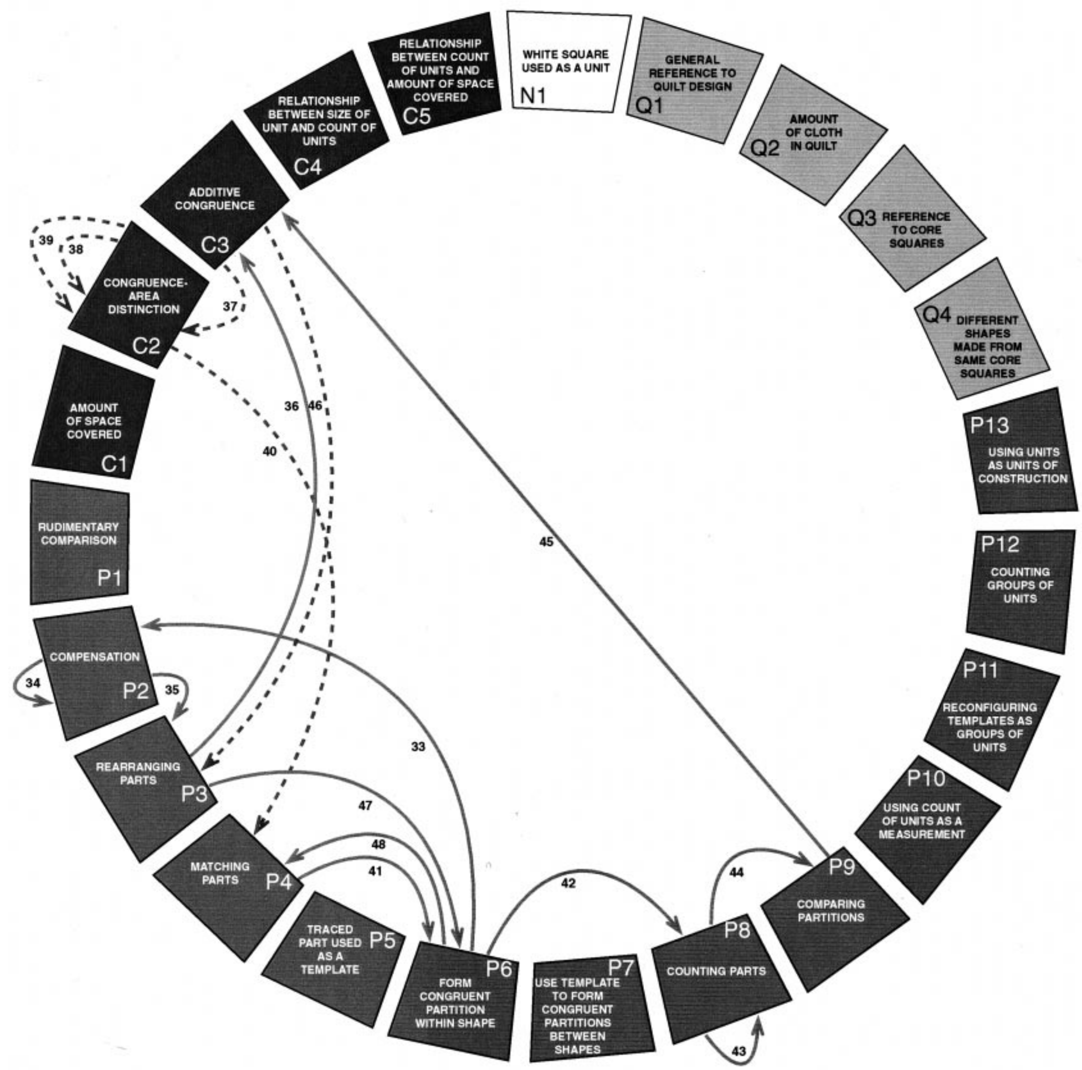


Fig. 2. Graph of segment 2 of the lesson.

2.9. Segment 4: units of measure emerge

2.9.1. Overview

This segment began with a redescription of the previous procedural sequence in which students invented templates. Then a student offered a procedural sequence using a different template (one third) to create a congruent partition between shapes A and C. The class compared the two different partitions of these shapes, focusing on the different number of parts in the partitions. One procedural sequence involved four equal parts, and the other involved only three. A student explained that the difference in the number of parts was a result of the different sizes of the templates.

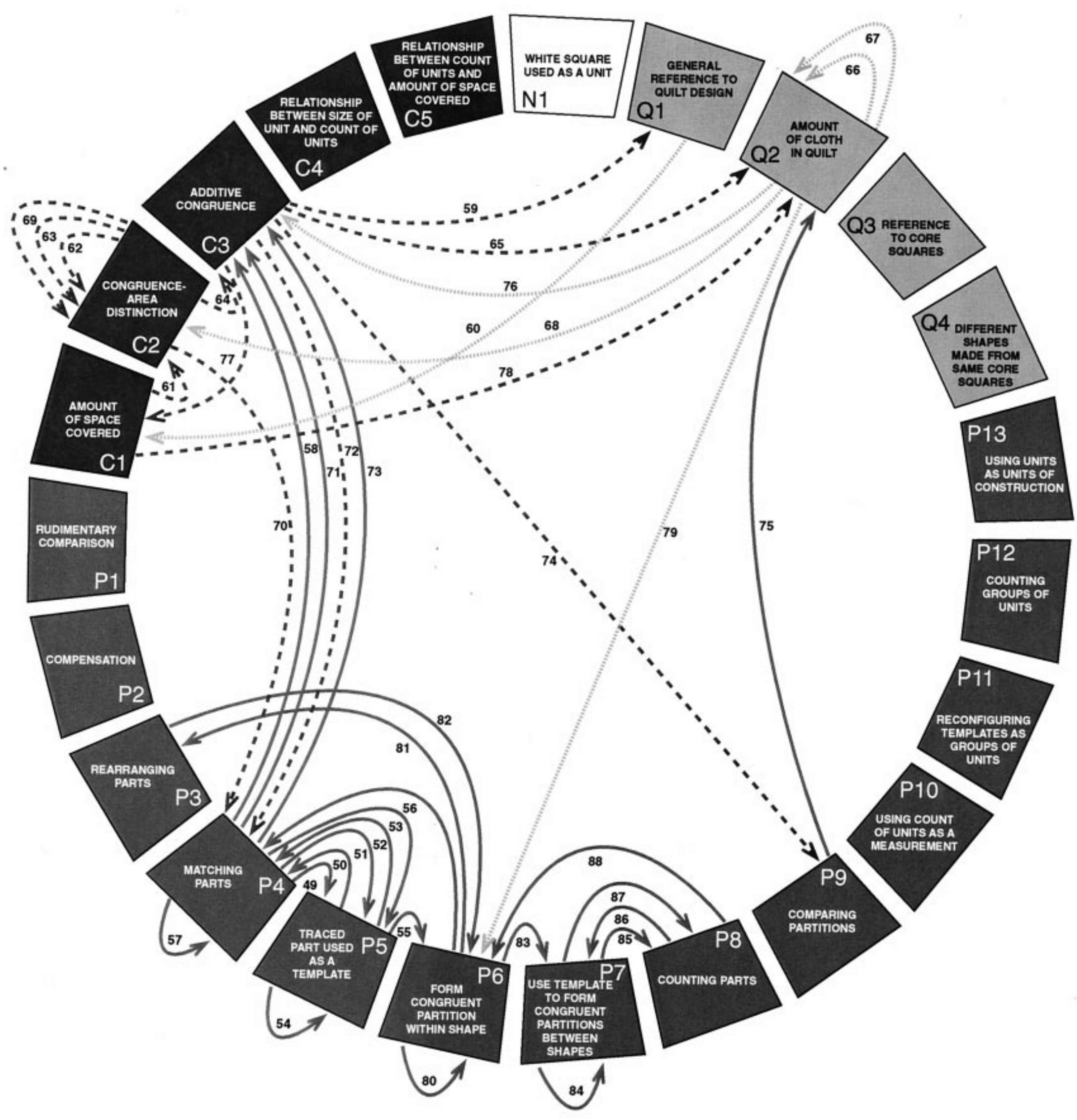


Fig. 3. Graph of segment 3 of the lesson.

During these two procedural sequences, shape C was folded into fourths and then into perpendicular thirds. After the partitioning into thirds was completed, a student unfolded the shape and counted the twelve squares formed by the intersection of the folds. One student conjectured that these twelve squares would cover shape A. The teacher asked students if these squares reminded them of anything, and several recalled, "core squares!" This was a reference to quilting, where a core square was a unit square that served as the building block in the particular method of quilt design explored by the class. At this point, the teacher introduced a separate white square. After congruence between the white square and one of the squares marked by the folds on shape C was established, the white square was used as a unit of measure. This new notation lifted the physical instantiation of the unit from particular parts of the rectangles and reified it as a mobile unit of measure, one connected both to previous history with the quilts and to current experience with the rectangles.

The class then explored how they would fit twelve of these squares into each of the shapes. Several students offered efficient methods of counting the number of white squares that would fit into the shapes; they described how to reconfigure the templates they had previously used to partition the shapes into groups of units, and then skip-counted these groups. For example, a student said that one of the congruent partitions would have "3 squares in each row, so that's 3, 6, 9, 12 squares."

2.9.2. Tracing the argument

Fig. 4 shows that the amount of mathematical activity in this segment (the number of codes per minute) nearly doubled compared with the previous segment. Fig. 4 also indicates a drastic shift in the nature of procedures employed. Only procedures involving congruent partitions, units, counting parts, and comparisons of partitions were used in this segment. The class established explicit ties between ideas about forming congruent partitions within shapes and units of measure (references to core squares). The notation of the core square, drawn from previous history in which core squares were used as base elements to design quilts, was developed and established as integral to the procedure of counting units to measure the area of a shape. Hence, in this segment history went beyond its initial role as a context for current activity, to enter into a specific form of mathematical practice, creating the unit—which, in turn, supported a new sense of area as a quantity based on a count of units.

2.10. Segment 5: summary and reflection

2.10.1. Overview

During this segment, the teacher asked the class to reflect on what they had learned that day. She made a particular effort to engage students who had not yet spoken during the lesson, asking them to volunteer to "talk to the group about what we found out today, what we've been thinking about. What was interesting to you that we found out today?"

During this discussion, students shared several procedural sequences and drew a few connections to additive congruence. The teacher reminded them that at the beginning of the lesson, they had not known that the three rectangles took up the same amount of space, because the shapes looked very different. Several students referred to procedures of partitioning and matching parts, and to using the count of units as a measurement to justify their conclusion that the shapes covered the same amount of space. The children continued to offer new procedural methods to show that the shapes covered the same amount of space until the teacher signaled the end of this discussion.

2.10.2. Tracing the argument

Fig. 5 shows that in this segment, children were recapitulating key parts of the argument developed to this point. However, the concepts were now connected to procedures of forming

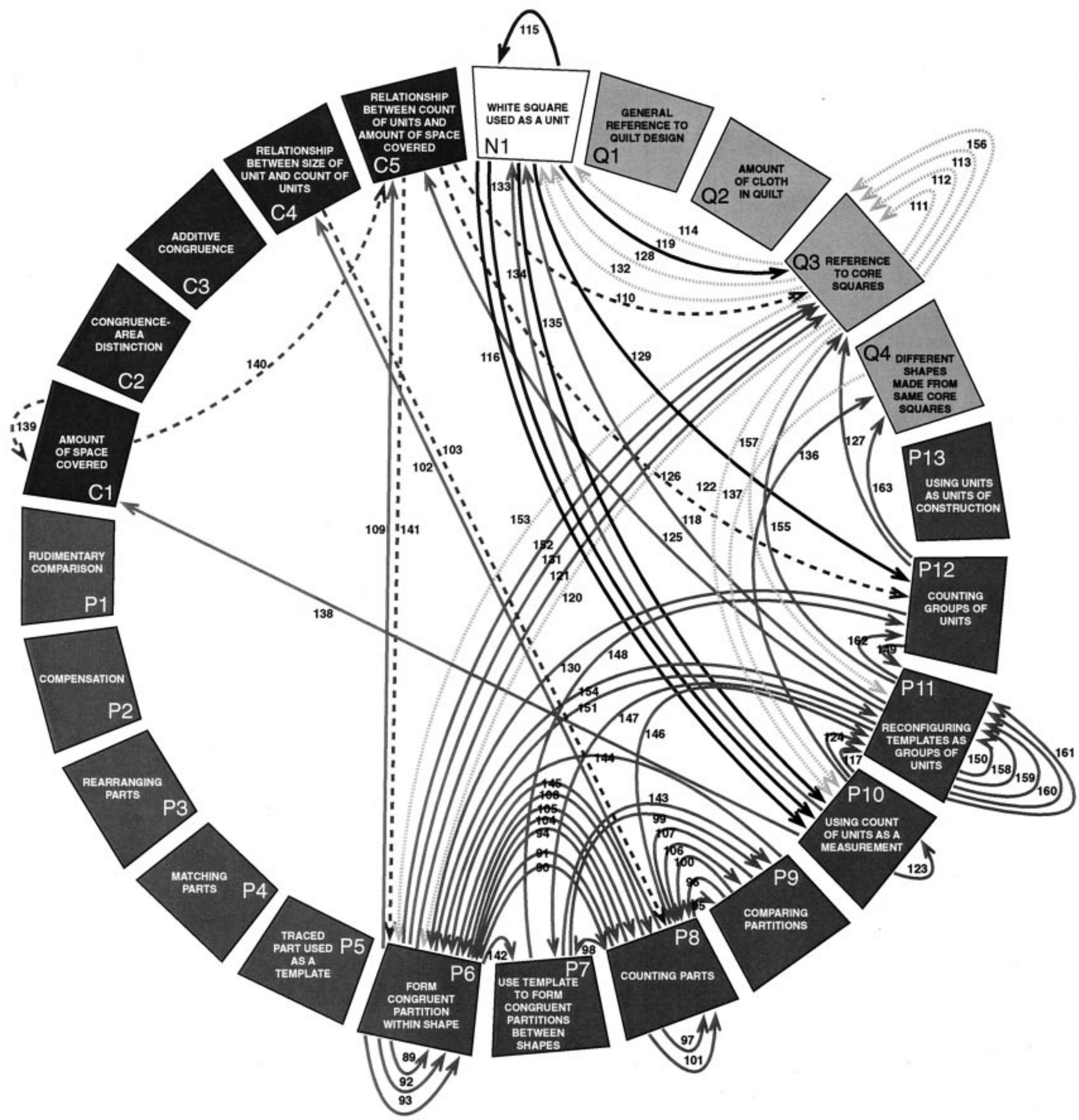


Fig. 4. Graph of segment 4 of the lesson.

congruent partitions, counting parts, and comparing partitions. When they presented procedural sequences not previously demonstrated by their classmates, the students relied on the more rudimentary procedure of matching parts, but they combined this matching of parts with the more sophisticated procedures of comparing partitions and counting of parts. These relations had been established tacitly, but not explicitly in the previous segments. In addition, some students used templates to form congruent partitions between shapes, and others proceeded to reconfigure these new templates into groups of units.

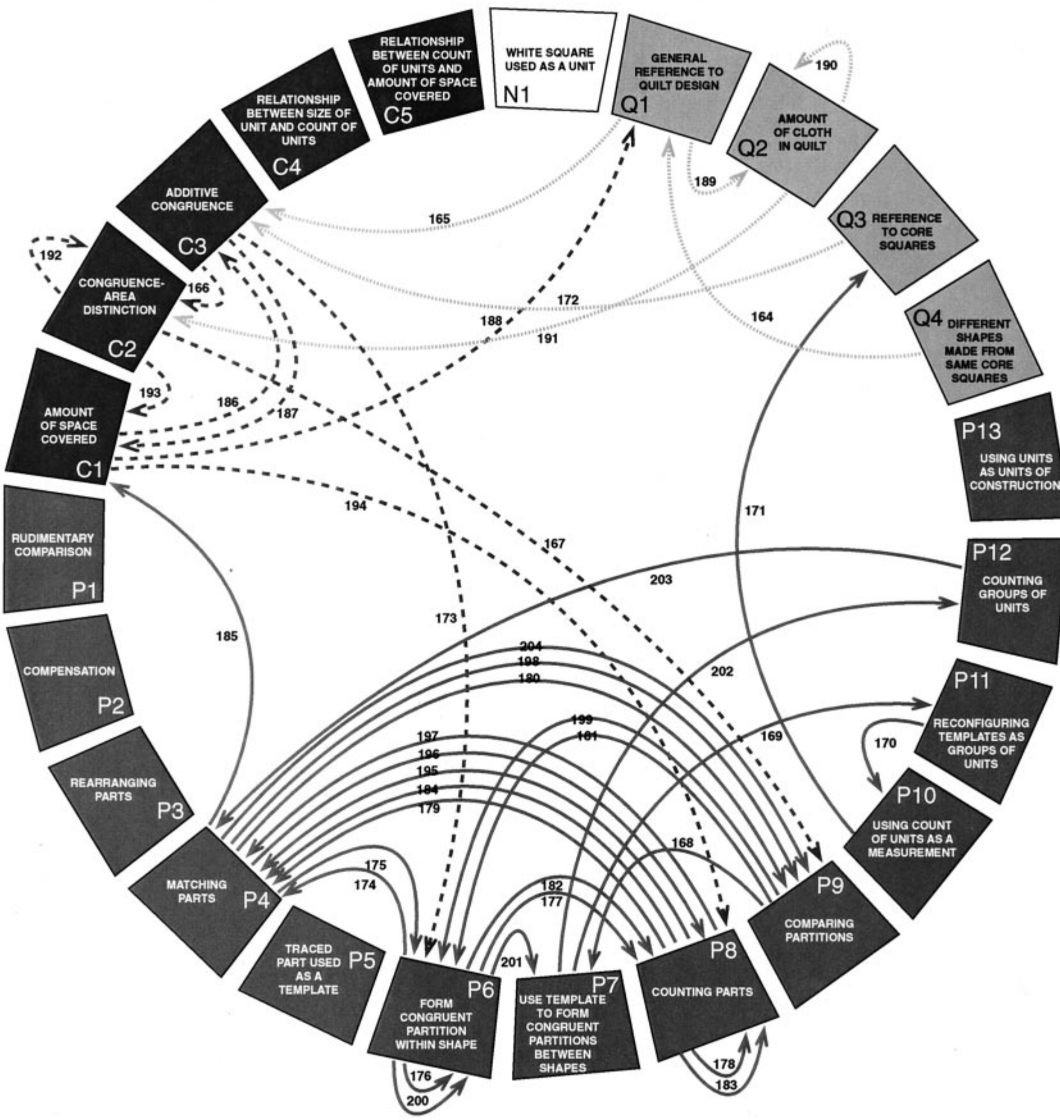


Fig. 5. Graph of segment 5 of the lesson.

2.11. Segment 6: introducing the construction task

In this brief (2 min) segment, the teacher introduced the next task. Students were asked to use computer quilt design software to create rectangles like the three displayed on the board. She instructed students to use 12 core squares (generated by the software) to create one rectangle, and then to use the same 12 squares to create other rectangles. The students shared their thoughts about the rectangles that would be easiest to build from the core squares, given the procedures of grouping, copying, and translating afforded by the software. During this discussion, students referred back to the templates and congruent partitions the class had

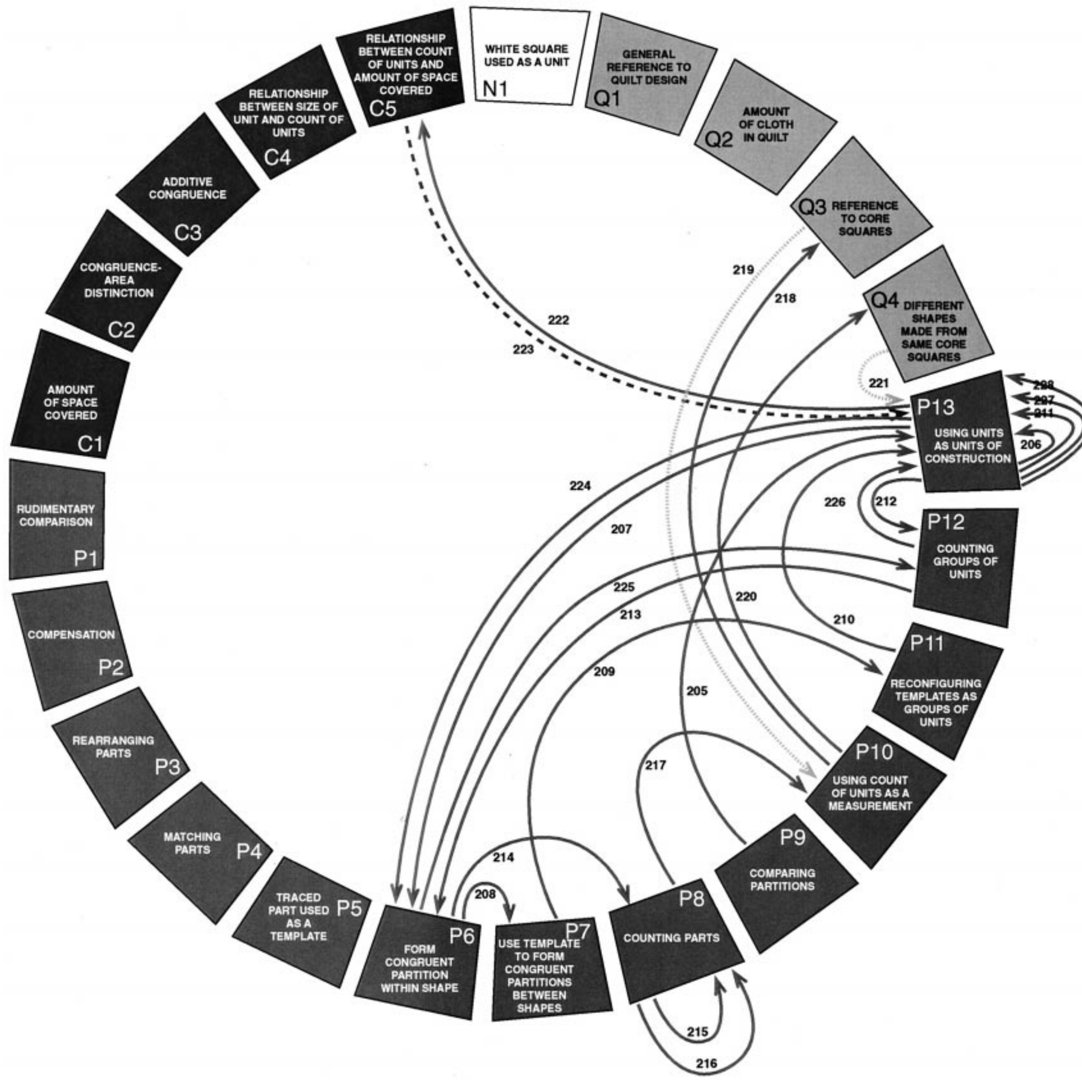
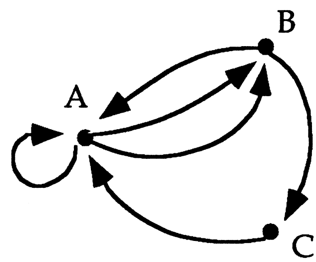


Fig. 6. Graph of segment 6 of the lesson.

used earlier and reflected on how this task would be much easier at this point than it would have been before all of the talking and thinking they did during the lesson. Finally, the teacher closed with the following challenge: “Can you use those same 12 core squares to make a different shape. . .? We’re going to find out how many different ways we can do that.” The flow of this extension of the argument is displayed in Fig. 6.

3. Exploratory data analysis of the graph

Displays of graphs like those in Figs. 1–6 convey the ebb and flow of points of argument during the lesson. Due to the purposeful arrangement of nodes in the graph, regions traversed



Initial / terminal ↓ / →	A	B	C
A	1	2	0
B	1	0	1
C	1	0	0

Fig. 7. A directed graph and its adjacency matrix.

during lesson segments illustrate the continual interplay between conceptual and procedural knowledge that accompanies the progressive emergence of more (mathematically) sophisticated senses of area. One can readily see that more sophisticated concepts are coupled with the emergence of more refined procedures, and that both are facilitated by students' access to prior history with the core squares used to design quilts.

For further analysis, we rely on matrix representations of these directed graphs. Here we explore one of the possible matrices that could be associated to the graph, the adjacency matrix. The adjacency matrix associated with a graph is obtained in the following way: First the vertices are written as the labels for the rows and columns in a square matrix (e.g., if there are three nodes labeled A, B, C, the corresponding square matrix will be of order three, with the rows and columns labeled A, B, C). Next, the edges of the graph are represented as the entries in the matrix. For a directed graph, the rows represent the initial vertex (starting node) of the edge, and the columns represent the terminal vertex (ending node) of that same edge. Thus a *1* in the cell with row A and column B would represent an edge that starts at vertex *A* and ends at vertex *B.* In a graph with multiple edges between two nodes, the number in the cell corresponds to the number of edges that start at node A and end at node B. Fig. 7 displays a simple directed graph and its adjacency matrix.

The adjacency matrix retains the directedness and count of edges on the graph, but not the global ordering of connections. Thus, in moving from the graph to the matrix representation of the discourse flow, some information about sequence of the flow is lost. Effectively, we collapse the connections across time but retain all information about which nodes are connected and how many connections were made between each pair of nodes during the course of the conversation. We assume that a similar process may occur in memory, especially over time. That is, individuals may not retain the precise sequence of an argument,

Table 4
Adjacency matrix for the lesson collapsed by semantic category

	History	Concept	Notation	Procedure	Out-Degree
History	9	7	3	10	29
Concept	5	19	0	17	41
Notation	1	0	1	4	6
Procedure	13	15	2	123	153
In-Degree	28	41	6	154	

but may well retain relationships among nodes. What remains is the gist of the argument—its structure.

The complete adjacency matrix for our classroom discourse graph is displayed in the Appendix. This 23×23 matrix summarizes the total number and the directedness of the edges connecting the nodes (representing the codes displayed in Tables 1–3) throughout the entire 52 min of the lesson. Empty cells should be considered as cells with zeros in them, as is the custom when dealing with sparse matrices. Simple counts in each cell indicate the number of transitions from one code to another (the diagonal represents self-reference).

In addition to the standard rows and columns, the matrix shown in the Appendix includes two summation rows and columns. These simple matrix computations yield information about the frequency of particular codes within the flow of discourse. The *in-degree*, located in the next-to-last row of the matrix, is simply the sum of the entries in each column, and represents the number of edges leading to a particular node, which in turn, indexes the number of times that code was referred to during the conversation. For example, code Q2 was referred to 13 times over the course of the lesson. Similarly, the *out-degree*, located in the next-to-last column of the matrix, is the sum of the row-entries, and represents the number of edges leading out of a particular node. (Since the flow of discourse is continuous, every edge into a node is followed immediately by an edge out of that same node, so in our graphs, the in-degree is necessarily equal to the out-degree for all nodes except the beginning and ending nodes of the graph.)

Indices of in- and out-degree summarize grosser points of the ebb and flow of conversation. Table 4, for instance, displays a block version of the matrix displayed in the Appendix, with rows and columns collapsed into (i.e., summed across) categories according to the mathematical senses (i.e., procedures, concepts, historical reference) that guided our analysis. The in-degree, located in the final row of the Table, indicates the number of edges leading to a block of nodes. For example, quilt history was referred to 28 times. The out-degree, located in the final column of the Table, indicates the number of edges emanating from a block of nodes. The diagonal elements of Table 4 suggest that conversational transitions often demonstrated chaining within a category, where, for example, one procedure suggested another. The off-diagonal elements indicate first-order relationships among mathematical senses that were established in the conversation. These relationships often consisted of exploring the implications of procedural means for related mathematical concepts or for anchoring a procedure by referring to previous history in the classroom.

Table 5
Centrality values for most highly connected nodes

Node	Brief Description	Value
P6	Divide into congruent parts	.52
P8	Count congruent parts	.39
P4	Part by part congruence	.37
C3	Additive congruence	.37
P11	Replace templates with units	.35
C1	Space covered	.35
Q3	Reference to core squares	.28
P10	Counts of units as measure	.26
P12	Count groups of units	.26
P9	Compare size of different partitions	.24

In addition, we were interested in the connectivity of the graph, indicated by the number of distinct nodes that are connected to any given node. Thus, for each node we calculated both *in-connections*, the number of distinct nodes connecting to that particular node, and *out-connections*, the number of distinct nodes emanating from that particular node. Unlike the in-degree and out-degree, the in-connections and out-connections often have different values for a given node. It is possible, for example, for one node to accept edges from many nodes, but for every edge that leaves that node to lead only to one other node. Thus, the in- and out-connections provide indices of the degree of connectivity of the graph. We employed these indicators of connectivity to identify the most central features of the content and structure of the argument. They represent a schema-based view of what might be represented in memory as a result of participating in this classroom conversation.

3.1. What is central to the argument?

We captured the gist of the argument by computing the centrality of each node of the graph. Centrality indicates a node's degree of participation in the structure of the graph by measuring the relative connectivity of a node within the graph. Nodes with relatively high centrality are vital to the structure of the graph; if they are removed from the graph, the remaining partial graph has fewer interconnections and less cohesion. Thus, nodes with high centrality values function as landmarks or *attractors* in the argument structure defined by the graph.

3.1.1. Measure of centrality

We calculated the centrality of each node as the sum of distinct nodes with connections to and from that particular node (in + out connections) divided by twice the number of nodes (46), so that the maximum value of the index would be 1.00. The ten most-central nodes for the complete adjacency matrix are displayed in Table 5. Generally, procedures establishing equivalence of area by partitioning were most central. Forming congruent partitions (P6 in Appendix) was the most central node in the graph, with more sophisticated (P8, counting congruent parts, and P11, reconfiguring congruent partitions as units) and less (P4, matching parts) mathematically sophisticated procedures as close counterparts (i.e., also high measures of centrality). Concepts with high centrality values included ideas of space covered (C1) and additive congruence (C3). These indices of connectivity accord well with our sense of how the conversation proceeded in real time and thus constitute a form of face validity.

3.1.2. Predicting recall

We explored the predictive potential of our model by comparing how the connectivity of the graph for the earlier portion of the lesson related to the structure of student recall during the latter portion. The fifth segment of the lesson, which we refer to as the *reflective segment*, was dominated by recapitulation, with the teacher asking students who had participated less (or not at all) in the preceding discussion to summarize salient points or to note what they found most interesting.

We conjectured that these students' reflections should be influenced by the structure of the argument earlier developed. In particular, the relative connectivity of nodes in the first four segments would predict the content and structure of their recollections. For this analysis, we computed the centrality of each node in the collective graph of the first four segments and compared these values to the centrality of the corresponding nodes in the fifth segment. We found that the centrality of nodes in the reflective segment was substantially related to the centrality of nodes in the earlier discussion, with correlation coefficient $r = 0.78$. The most central nodes in the reflective segment corresponded to increasingly sophisticated ways of recruiting partitioning (e.g., matching parts, forming congruent partitions, and counting congruent partitions) to persuade others that the three rectangles all had the same area, and these senses were also most central in the preceding segments of the lesson. In addition, the concepts, procedures, and historical references corresponding to nodes with lowest centrality in the first four segments were not mentioned at all in the reflection segment (i.e., the nodes with lowest centrality in the first four segments had zero centrality in the reflection segment). Thus, the content and structure of the earlier argument, as measured by centrality of nodes, was predictive of the content and structure of the students' recall of the argument.

3.2. Further explorations of relationships

Raising the adjacency matrix to the nth power, where n is any positive integer, yields a matrix with entries that correspond to n-step connections between nodes. In this way, it is possible to identify relationships among nodes that are indirect. We adjusted the adjacency matrix to eliminate self-reference, setting the diagonal elements to zero. Consequently, squaring this adjusted matrix (the results of which are displayed in the Appendix) indicates the number of 2-step connections between any two nodes. Consider, for example, the node referring to counts of units as a measurement (P10 in Appendix). We found that this node was reachable in two steps from/to all but seven of the other nodes. The number of two-step connections with this node was highest for the history node that referred to the use of core squares in quilting, to the procedure of forming partitions within a shape, and to the procedure of reconfiguring the templates into counts of units. This node was also reachable via other nodes (a count of 11 self-references in the matrix), suggesting again the importance of this procedure to the resolution of the argument. The widespread reach of this node suggests that although acts involving units were developed late in the argument, once they were in place, they played an integrative role in the structure of the lesson.

In summary, elementary matrix operations on adjacency matrices associated with the graph of the classroom conversation provided further insight into the structure of the argument. The adjacency matrix summarized associations between nodes. Measures of centrality showed which nodes participated most in the development of the argument during the first forty and final ten minutes of the lesson, the reflective segment. Although the centrality of procedures involving partitioning was not explicitly marked in conversation during the first forty minutes of the lesson, it was clearly constructed by students, because their reflections about the lesson had the same structure. Thus, measures about centrality derived from the graph proved predictive of what students found most salient about the gist of the argument. Other forms of exploratory analysis relied on indirect relations (reachability) between different concepts and procedures. The act of creating units, occurred relatively late in the lesson, but it was often related within two steps to other procedures, conceptual senses, previous history, and notation.

4. Orchestration of a collective argument

We turn next to developing a complementary analysis of the classroom discourse that describes how the teacher helped children formulate the structure of the argument that we have displayed as a graph. Our interest in this teacher's assistance stems, in part, from images of teaching as assisted performance (e.g., Tharp & Gallimore, 1988) and in part, from images of learning as guided or as "social" participation (e.g., Rogoff, 1990; Wenger, 1998). In these images of teaching and learning, classroom teachers socialize their students into the practice of mathematics (e.g., Arcavi, Meira, Smith, & Kessel, 1991). Yet socialization can occur in a variety of ways; we emphasize how the teacher employs the turn structure of conversation to orchestrate collective argument (Forman et al., 1998; Krummheuer, 1995).

O'Connor and Michaels (1996) argue that teacher orchestration "provides a site for aligning students with each other and with the content of the academic work while simultaneously socializing them into particular ways of speaking and thinking" (p. 65). O'Connor and Michaels (1993; 1996) further suggest that proficient orchestration relies on teacher revoicing (e.g., expanding, refining) of student talk. The notion of revoicing derives from Goffman's (1974; 1981) proposal that in most conversations, even those that are ostensibly two-person dialogues, there is, in fact, an expanded repertoire of roles beyond that of speaker and listener. Goffman suggests that the speaker in a conversation takes at least three roles: an animator (the person making the utterance), an author (the person scripting the talk), and a principal (the person whose position is being represented). For example, in a political speech delivered by the President of the United States, the animator (the President or perhaps his spokesperson), is unlikely to be the author of the words (or the sole author). Moreover, the animator may not be the principal (for example, the President may give a speech representing the position of his Secretary of State).

When a classroom teacher revoices her student's words, she is acting as the animator while the student is the principal (and perhaps the author). For example, a student may explain how she solved a perimeter problem by saying that she counted all around the hexagonal shape. In response, her teacher might rephrase the student's utterance by substituting "perimeter" for her expression "all around." In this instance, the teacher is substituting a mathematical term, "perimeter," for a more familiar, but imprecise construction, "all around," thereby transforming the student's utterance spoken in everyday language into mathematical reference. The teacher thus rerepresents the position of the principal (the student who said "all around").

We analyzed the classroom teacher's revoicing of student talk with an eye toward understanding how the resulting expansions of conversational roles might account for the emergent structure of the argument analyzed previously. Of course, revoicing encompasses more complex goals than substitution of mathematical vocabulary for everyday words, or even expanding the range of a mathematical concept. Some revoicing appears to be aimed at communicating respect for ideas and at the larger agenda of helping students identify with the serious and important business of mathematical activity (imagining, inscribing, arguing, valuing mathematical ideas, and solving problems). Thus, we employed Goffman's analytic framework to understand how the teacher amplified and transformed the mathematical power of student talk and also, how she invited students to identify with this form of activity. Our structural analysis suggested that the emergence of unit was a particularly well connected portion of the argument (see Segment 4). Letting the structural analysis drive our decisions about how to deploy our conversational analysis, we focus on the discursive means by which the teacher, Ms. Curtis, orchestrated this emergence of unit.

We analyzed several passages from the fourth segment of the classroom conversation. In each passage, we suggest how Goffman's notions of participant frameworks function to clarify how Ms. Curtis brought into contact the different mathematical senses displayed in that segment of the graph (i.e., different ideas about the nature of the space covered, mathematical procedures, and prior experiences). Bringing into contact has a structural interpretation in the graph as increasing connectivity, but as we shall illustrate, this connectivity is a product of the conversational moves engendered by the teacher.

In the first passage, students generate two different congruent partitions of the rectangles (thirds and fourths) and test their utility for establishing additive congruence ("making C into A"). Ms. Curtis skillfully raises the conceptual ante by asking how two such different partitions could possibly have the same result. As a consequence, these forms of procedural knowledge are further refined by explicit comparisons between these partitions, along with attention to a conceptual sense of area as a count of units. In the (second) passage immediately following, Ms. Curtis draws collective attention to one student who has merged the two forms of partitioning (one vertical, one horizontal) into an array. This sets the stage for introducing a privileged partition, a square. Ms. Curtis guides the conversation toward the procedural implications of this new method for area measure and again relates this privileged partition to the concept of recruiting units as measures of area. Finally, in the third passage, Ms. Curtis orchestrates further connectivity among senses of area by drawing attention to a previous history of a square as a unit of composition and dissection in quilt design. The result is a conversational sequence that integrates a notation of square with procedures for counting groups of squares, and with a concept of area measure as a relation between a count of units and amount of space covered. However, as the analysis makes clear, the conversation does much more than merely draw these connections. The conversational space also creates room for students to ratify identities as mathematical participants.

The analyses which follow are based on detailed transcriptions of this part of the lesson. The transcription conventions used are the following: phrases enclosed in brackets contain descriptions of the nonverbal context of the discussion; each turn at talk is given a different number in the left-hand column of the transcript; punctuation is determined by intonation as well as by grammatical structure; when a student speaker cannot be identified, his or her name is replaced by *student* or *students* in the second column; the teacher's turns are identified by her name, Ms. Curtis.

4.1. First passage

The immediate prior context for this part of the lesson involved students folding and superimposing parts to establish that Shapes A and B "covered the same amount of space." Then, similar procedures were tried with Shapes B and C. Finally, the class began a comparison of Shapes A and C. One student, Mike, proposed folding Shape C into four equal rectangles (along the long side of C) and showed how one could superimpose each of those rectangles on Shape A so that one quarter of C overlapped one quarter of A. That demonstration was performed in the front of the class by Mike and Ms. Curtis. Then, Tim proposed a second procedure for comparing Shapes A and C: folding Shape C into three equal rectangles (along its short side) and then superimposing this third of C onto A. Using Tim's procedure, one-third of Shape C can be shown to cover one-third of Shape A. At the beginning of this first passage, Tim was standing near the teacher in the front of the room as she folded Shape C as per his instructions. The teacher animated Tim as author of the procedure of partitioning by thirds, after first ensuring that he understood "thirds" as three congruent partitions. In the first exchange immediately after this episode, the following discussion took place (To illustrate connections between our two levels of analysis, we also mark each turn in this first passage with the code, if any, generated in the previous analysis. A brief rationale for each code is enclosed in parenthesis):

[Tim stands at the front of the room with Ms. Curtis who is folding Shape C into thirds.]

Stephanie: (Turn 01) I can see that you can make anything out of A.

Ms. Curtis: (Turn 02) What do you mean you can make anything out of A?

Stephanie: (Turn 03) Like you can make B into A, and you can make C into A, but I don't know how to do that because it has to be four strips. *P8,* Edge 95. (Reference to counting parts in partitions. Stephanie is puzzled because different-sized units yield different counts of units to cover the same space.)

Ms. Curtis: (Turn 04) No, Tim. –Oh, yeah but if you take four strips from Mike, it would only—Tim says it's only gonna take him three, and you don't know if that's gonna work? *P9,* Edge 96 (Explicit comparison of partitions) *P8,* Edge 97 (Count partitions).

Stephanie: (Turn 05) Oh, now I see. [Watches as Tim places the folded third of C over A.]

Student: (Turn 06) Yes it will work, but it will take three, or three and a half. *P8,* Edge 98 (Count partitions).

[Tim attempts to place the folded shape C along shape A, but doesn't mark the endpoints.]

Ms. Curtis: (Turn 07) Maybe I'll put my finger where this one ends.

[Tim places folded C along A, with Ms. Curtis' finger acting as endpoints.]

P7, Edge 99 (Action creates template to form congruent partitions between shapes.)

Student: (Turn 08) We already did it. *P9*, Edge 100. (Reference to comparing partitions.)

Student: (Turn 09)No, not with three. *P8,* Edge 101. (Reference to counting parts in partitions.)

Ms. Curtis: (Turn 10)Now how could that be? Like Stephanie is saying. It took Michael four strips to cover A. *P8,* Edge 102. (Question directs attention to different units/different counts.)

Stephanie: (Turn 11a) I know. [Waves her hand in the air while she speaks.]

Stephanie: (Turn 11b) You have to use the long side. The long sides, because the long sides are bigger than the short sides. *C4,* Edge 103 (Connects smaller count to bigger unit.)

This passage begins with Stephanie asserting something about the generative power of Shape A: that anything could be made out of it. She marks her turn as the author, "I can see" but ends that turn with assignment of the principal as the more impersonal "you," implying the generality of the claim (anybody could). Nevertheless, her utterance is ambiguous, even in this situation where they have been comparing the three shapes. Ms. Curtis requests disambiguation of her claim by revoicing Stephanie's statement that "you can make anything out of A," thereby preserving the impersonal voice of the original utterance, but questioning its clarity (an important aspect of mathematical discourse, Forman et al., 1998). In the next turn, Stephanie complies with the request for clarification by summarizing the procedures employed to establish the equivalence of Shapes A, B, and C. At the same time, she expresses confusion about the procedural relationship between Shapes A and C by asserting that you have to fold C into four pieces to overlap it with A (not the three pieces that Tim has claimed will also work). In addition, Stephanie maintains her orientation toward collective claims but switches to the more emotive voice of the author: "I don't know how to do that." Thus, her utterance aligns her simultaneously with a collective principal and a personal identity.

Ms. Curtis responds to Stephanie's confusion by asking about its nature: Is she wondering how two solutions (Mike's and Tim's) could both be correct? Simultaneously, the teacher attributes different solutions to the two students and aligns them with different positions in an argument. Thus, while she does not repeat or expand Stephanie's previous utterance, she recasts it as a contrast between two positions. In this way Ms. Curtis animates Stephanie's interpersonal voice in the classroom discussion. Through this form of revoicing, she articulates Stephanie's summary of the debate and maintains Stephanie's ownership of the claim. In Goffman's terms, Ms. Curtis is the animator, Stephanie is the principal and the author of the explanation.

As Tim begins to demonstrate his procedure by superimposing Shape C folded into thirds over Shape A, Stephanie asserts that she understands the meaning of his actions ("Oh, now I see"), again identifying with the actions. In the next turn, another (unidentified) student expresses at least partial agreement with Stephanie's implicit support of Tim's procedure for establishing the equivalence of the two shapes. The student revoices Ms. Curtis by rephrasing her utterance, "gonna work," to "it will work." Nevertheless, this student is not sure whether folding Shape C into thirds will exactly cover Shape A (it might take three or three and a half). The statement illustrates that the class's shared understanding of these folding and overlapping procedures is still fragile. Some students, like Tim and perhaps Stephanie, are willing to accept the possibility that different partitions of C could cover A. To accept that solution, others may need to observe the procedure being applied. These students are liable to be misled by poorly-specified procedures (such as failing to mark clearly the end points of the partition as the folded Shape C is overlapped with, and then slid across the face of Shape A). Therefore, Ms. Curtis intervenes so that the endpoints of each placement of Shape C are carefully indicated, in this case, with chalk. She thus inscribes what is mathematically important, highlighting the iteration implicit in the procedure. Instead of doing the demonstration for him, Ms. Curtis assists Tim as he demonstrates his position in the collective argument. In this way, she supports the articulation of his position rather than seizing it from him.

Two other (unidentified) students express the class' continuing difficulty with keeping track of the two opposing argumentative positions being played out in front of them (Mike's four fold solution vs. Tim's three fold solution). The first student expresses confusion about the originality of Tim's procedure ("We already did it"), whereas the second clarifies the contribution of this new method ("No, not with three"). Here we see students taking an active role in listening to each other's concerns and attempting to address them.

After Tim finishes his overlapping procedure, Ms. Curtis asks the class to reflect on the results of their two solutions ("Now, how could that be?") and, using an indirect quote, revoices Stephanie's previous summary of Mike's earlier solution. In this turn (10), Ms. Curtis animates the classroom community's perspective on the results of these two previous procedures. She appeals to curiosity and the need for coherence by questioning the intelligibility of the two solutions: Could both be correct? The question follows from the stance of a mathematician who searches for invariants and who appreciates the importance of unit in measure, although this perspective is not necessarily shared by all the students. In addition, Ms. Curtis animates Stephanie and Mike by expressing a position associated with both of them: that Shape C can be folded into four pieces and be shown to cover Shape A. Thus, Ms. Curtis is the animator, but Mike is the author and the principal. Stephanie is also a principal.

In response to Ms. Curtis' question ("How could that be?"), Stephanie clarifies her understanding of how Shape C could be folded into only three pieces and still cover Shape A. She articulates how this seemingly anomalous solution could work: folding on the long side of Shape C would naturally result in a different result than folding on the short side of Shape C. In this turn, Stephanie not only justifies both results, but also moves from a procedural explanation to a conceptual one. That is, by dividing a longer distance by a larger number one may get the same result as dividing a shorter distance by a smaller number. Stephanie animates a collective principal by again making this claim by employing the impersonal form of you in the present tense. The form of the statement also tacitly assumes that there is no need now to explicitly address anybody. This lack of explicit address suggests that she assumes that this claim can now "be taken as shared" (Cobb et al., 1993).

4.2. Second passage

The second passage begins with a shift in focus from the comparison between Mike's and Tim's solutions, to a new topic that has emerged as a result of folding Shape C twice, along the two different dimensions. The two sets of folds have produced a 3×4 array of folded squares on Shape C: a result that had been anticipated by Ms. Curtis when she designed this task. Instead of pointing out this result, she watches as the students discover it for themselves. The first discovery occurs while Stephanie takes her conversational turn (#11), discussed above. As he unfolds Shape C, Tim sees the squares appear and counts them in front of his classmate, Matt. Ms. Curtis watches and laughs. Tim is standing in the front of the class but off to one side. This discussion follows:

Ms. Curtis: (Turn 12a) Tim, do that again.

Ms. Curtis: (Turn 12b) [Takes Shape C from Tim and holds it up in front of her.] *Watch.* Do that again.

Ms. Curtis: (Turn 12c) No, not this. What you were just doing in front of Matt? Do that again.

Tim: (Turn 13) One, two, three, four, five, six. . .

Ms. Curtis: (Turn 14) Uh-huh, do that.

Tim: (Turn 15) One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve [points to each square as he counts it]

Ms. Curtis: (Turn 16) Twelve what?

Tim: (Turn 17) Twelve squares.

Stephanie: (Turn 18) That makes it twelve.

Becky: (Turn 19) You folded it to make. . . [Points to Shape C]

Tim: (Turn 20) [Points to each row on Shape C] 3, 6, 9, 12 squares

Ben: (Turn 21) Oh, then it takes twelve squares to make that. [points to A]

Ms. Curtis: (Turn 22) To make this? [Points to A?]

Ben: (Turn 23) Yeah.

Stephanie: (Turn 24) [Stands up and points to C] Because you folded it.

Ms. Curtis: (Turn 25a) Just a second. [Folds up C] Let me fold it up so it's just showing one of the squares Tim was counting for you. We're gonna look at these first.

Ms. Curtis: (Turn 25b) Hold on, hold on. Three, two, one, zero. [Normative directive for quiet.]

Ms. Curtis: (Turn 25c) Ben says that if this took twelve squares to make – I folded it down, so now you're only seeing one of those squares

Ms. Curtis: (Turn 25 days) Ben says that if the way we've been thinking is really right, then A and B [corrects to say] A and C are the same size – that twelve of these should fit into A.

Ms. Curtis highlights this creation of a unit of measure by asking Tim to repeat his actions so that everyone in the class can see them ("Tim, do that again"). She expands her request by taking Shape C from Tim, holding it in front of her, explicitly commanding the class to watch, and repeating her request to Tim. Initially, Tim seems confused about what he is supposed to repeat. The significance of the squares may not be apparent to him at this point. He tries to take Shape C from Ms. Curtis, but she does not give it to him. Seeing Tim's confusion, Ms. Curtis clarifies her request by asking him to repeat what he just did in front of Matt. Tim appears to understand this time, because he begins counting the array of squares. Ms. Curtis encourages him to continue counting.

When he finishes, Ms. Curtis revoices the final number and asks for further expansion ("Twelve what?"). Tim replies, "Twelve squares." Stephanie revoices both Ms. Curtis and Tim but does not add additional mathematical content to their utterances. Becky chimes in by connecting the result of the folding procedures (the twelve squares) with the procedures enacted by Tim, and perhaps Mike (". . .you folded it to make. . ."). Tim completes Becky's utterance by specifying the outcome of the folding procedures (four rows of three squares each). Thus, Ms. Curtis and Stephanie both animate Tim's explanation. Both refer to the number twelve, first mentioned by Tim as the result of his counting. However, it is clear from the context that Tim is the author and the principal of this description of the results of his actions. Becky also participates in the animation of Tim's actions by explicitly referring to "you folded it to make," and Tim agrees by completing and clarifying her utterance. In this way, Ms. Curtis, Stephanie, and Becky all participate in elucidating and expanding Tim's explanation, while attributing the source of the explanation and procedure to Tim.

In the next turn, Ben articulates his understanding of what they have learned: Since Shape C is composed of twelve squares, and since Shape A has been shown to cover the same amount of space as Shape C, then Shape A must also be composed of twelve squares. Part of Ben's explanation is nonverbal. He points to, but does not mention Shape A. But Ben's explanation is not a mere summary of what has already been established. Instead, he draws a transitive inference (if $C = 12$ squares and $A = C$, then $A = 12$ squares) that contains the beginnings of a new understanding of area. This new understanding is the connection between the procedure of folding and overlapping to establish equivalence, and the concept of a unit of measure, which achieves the same result through a process of iteration. In response, Ms. Curtis acts as if she does not fully understand the implications of Ben's explanation. She asks him to disambiguate the referent of his inference (Shape A), and he does. Perhaps she feels the need to clarify the last part of Ben's explanation for the other students in the class. This is especially important because the last, crucial part of his explanation was not vocalized, but was delivered by nonverbal pointing. At this point, Stephanie joins the conversation by adding that Shape C was the source of the square unit when it was folded up (in two directions).

Ms. Curtis summarizes Tim's and Ben's contributions to the collective argument. First, she folds Shape C again and explains that she is doing so to show one of the squares that Tim was counting. In this way, she focuses the class's attention on the single unit, rather than the entire 4×3 grid. In turn 25, she quiets the class down so that she can be confident that everyone is paying attention. Next, she revoices Ben's explanation while displaying the single unit: If Shape A and Shape C are the same size, then the twelve squares that make up Shape C should also fit into Shape A. This expansion of Ben's (largely nonverbal transitive inference) by Ms. Curtis makes her an author of the transitive inference as well as Ben's animator. Nevertheless, the form of the expansion attributes the explanation to Ben (making Ben an author and the principal). Ms. Curtis also puts words into Ben's mouth when she says, "Ben says that. . . if the way we've been thinking is really right, then A. . . and C are the same size—that twelve of these should fit into A." Note the difference between Ben's actual explanation, "Oh, then it takes twelve squares to make that. [Points to Shape A]" and his teacher's expansion of his explanation. Ms. Curtis provides a clear, logical construction in words (using if, then) for Ben's implicit and nonverbal explanation, while still making it clear that the explanation belongs to Ben.

4.3. Third passage

The third passage is initiated by Ms. Curtis, who helps students establish correspondence between previous history and current activity. She repurposes a unit used in a previous lesson, a core square, to a unit of measure. The discussion follows:

Ms. Curtis: (Turn 25e) Is there anything familiar about this shape I'm holding right here? [Stephanie has had her hand raised throughout this turn but Ms. Curtis ignores her].

Stephanie: (Turn 26) I know. [Waves her hand in the air.]

Ms. Curtis: (Turn 27) Is there anything familiar about this shape I'm holding right here? Anything that looks. . . [Holds the shape by its corners in front of her so all can see.]

Student: (Turn 28) I know. [Other students also have their hands in the air].

Student: (Turn 29) It looks square.

Ms. Curtis: (Turn 30) . . . it's a square, but is there anything. . .

Ben: (Turn 31) Core square [Points to the square.]

Ms. Curtis: (Turn 32) [Opens her mouth in pretended amazement at Ben's answer.] What are you thinking, Rachel? [Rachel has had her hand up.]

Rachel: (Turn 33) A core square.

Ms. Curtis: (Turn 34) You've been working with squares this size for about three and a half weeks now.

Stephanie: (Turn 35) You can use the core square and copy the core squares. You can keep on copying them. . . . (?) [Stands up and using her hand as if it were the core square, she moves it along the length of Shape A. Becky takes the square from Ms. Curtis and stands up behind Stephanie, holds the square, and copies Stephanie's actions with the actual square.]

Ms. Curtis: (Turn 36) [Motions for Stephanie and Becky to sit down and picks up another square.] OK, so are these about the same size? [holds up white core square]

Student: (Turn 37) Yes.

Ms. Curtis: (Turn 38)Yes. So. . . . Ben says that twelve of them would fit in a strip. [Imitates Stephanie's and Becky's iteration of unit action along the length of A.]

Tim: (Turn 39) Six in each row, six in each row. [Stands up and gestures with both hands to right half of A and left half of A—two 1×6 parts of A] [Ms. Curtis stands watching him intently with the square over her mouth as if to silence herself, also nods to signal agreement with Tim's proposal.]

Ms. Curtis: (Turn 40) OK. Well then, if you're so smart, how many core squares could I fit into Shape B?

Student: (Turn 41) Twelve.

Ms. Curtis: (Turn 42) Twelve! [Looks astonished.]

At this point in the lesson, Ms. Curtis asks the class to relate the unit square to something else that they have studied in a previous lesson. Stephanie volunteers an answer, but Ms. Curtis does not call on her. Instead, she repeats her question to the class. Other students volunteer to answer and one reports that this shape looks square. Ms. Curtis agrees, but suggests she is looking for another response. Ben replies that it resembles a core square, a concept that they had used extensively in their previous work with quilts. Ms. Curtis' look of feigned surprise is a subtle way of expressing her approval of his answer and a means of heightening the excitement of discovering a link between the two lessons. Rachel also agrees with Ben's answer, indicating that she thinks she understands the meaning of Ms. Curtis' emotional response to Ben's reply.

Although Ms. Curtis does not explicitly evaluate Rachel's or Ben's response, she implicitly validates them by linking their responses, "core square," to the experience of making quilts. She also speaks in a personal voice to remind them of their collective history with quilts, animating a collective recall. Stephanie responds to the reminder by expanding Ben's and Rachel's utterances to include the procedure by which core squares can be used to measure the area of Shape A. Her language is less personal than her teacher's, since Stephanie's "you" is not any particular person and her present tense indicates a procedure that would operate at any time. She demonstrates her meaning through a gesture indicating how one can slide a core square across the face of a larger shape to iterate the unit of measure. Stephanie explicitly relates a transformation of translation from a previous lesson to iteration of a unit in the current lesson. Becky shows her agreement by copying the gesture with the square from the folded Shape C that she takes from Ms. Curtis. In this way, Stephanie revoices Ben and Rachel via both verbal and nonverbal means, and Becky nonverbally revoices Stephanie. Moreover, Stephanie and Becky demonstrate the connection between the sliding procedure that they had used when making quilts from core squares and the iteration of unit procedure that they are discussing now as a way to conceptualize area measurement.

Ms. Curtis takes the floor from Stephanie and Becky without commenting on their explanations. She continues the discussion of units of measurement by asking about the size of another square (not the square created by folding Shape C). Here Ms. Curtis shifts from the square created by their folding procedure to another "core" square (of equivalent size) that could be used as a unit of measurement. She asks them to accept the premise that the two squares are the same size, so that they can use the core square to iterate area, as well. The core square inscribes the concept and procedure of unit, unifying them and making them (literally) handy. This teaching move informs children about the perspective of the discipline. After several students acknowledge their agreement with her premise, Ms. Curtis revoices Ben's explanation and imitates Stephanie's and Becky's actions of sliding the core square across the face of Shape A to measure its area. In this turn, Ms. Curtis animates all three students: Ben through verbal means and Stephanie and Becky through nonverbal means. Ben, Stephanie, and Becky are presented as the authors and principals of this explanation of area measurement.

In the same turn, Ms. Curtis also adds the crucial piece of information about the numerical result of using the unit in this way—that it allows one to measure the area of Shape A. Tim responds by expanding the numerical part of his teacher's utterance. He adds that one-half of the area of Shape A would be the same as six units and the other half would also be six units. Thus, the area could be partitioned into two sets (or rows, in his terminology) of six units apiece. Tim elaborates this verbal utterance with a nonverbal gesture indicating the two partitions of Shape A. Ms. Curtis nods her acknowledgment.

Ms. Curtis then shifts the focus of the discussion from Shapes A and C to Shape B. This utterance is spoken in a playful tone of voice reinforced with her facial expression and her verbal reference to, “If you’re so smart.” From a disciplinary perspective, she asks that students generalize from their previous activities with all three shapes. She challenges them to show that what they’ve just learned about the iteration of a unit to establish the equivalence of two shapes (A and C), previously shown to be equivalent through the operations of folding and overlapping. Could iteration of a unit also be used to prove the equivalence of Shape B to A and C? In addition, she links the current activities to their previous history of using core squares to produce quilts. Once again, the juxtaposition of positive emotions and intellectual challenge is communicated, both verbally and nonverbally. Ms. Curtis animates herself as a principal participating in mathematical discovery (“How many core squares could I fit into Shape B?”), expressing emotions and teasing other members of the community, as well as offering challenges.

4.4. Summary of discourse analysis

Ms. Curtis mediated between the immediate, sensory world of her students and the hypothetical world of mathematics and mathematicians. She accomplished this mediation through a number of linguistic and psychological actions that collectively expanded the repertoire of conversational roles. During the course of the lesson, Ms. Curtis often outlined the positions in the argument, assigned students to positions, reminded students of inconsistencies in their rationales, clarified the argument as it was being developed, and supported student actions (e.g., acting as the placeholder to iterate the unit) so that mathematically valuable acts were highlighted. The discursive means by which she implemented these forms of assistance included expanding and refining student utterances, animating and attributing student authorship to ideas, and ratifying student speech for the collective. These forms of discourse were anchored within a more general participant framework. Ms. Curtis communicated her own personal investment in student thinking and established an atmosphere of excitement and surprise as the argument twisted and turned. This stance invited participation and identification with mathematics as an everyday activity. Revoicing student contributions, she also engendered discipline-informed ways of talking and acting even as she provided links to everyday thought and talk. She was particularly effective at helping children relate mathematical procedures (e.g., folding, overlapping) and concepts (e.g., unit), consistent with the structure observed in the directed graph. The combination of enduring classroom norms (e.g., be consistent, be precise) and moment-to-moment adjustments in the structure of student participation (e.g., animating student voices, attributing authorship, invoking particular students as principals) produced the collective capacity to experience mathematics as a language for making sense of the world.

5. Discussion

Mathematics educators are calling for expanded senses of teaching and learning mathematics, so that mathematical practices are interwoven with disciplinary knowledge throughout schooling (NCTM, 2000). This is a challenging agenda, one that creates significant dilemmas for educational designers. What forms of mathematics might children reasonably participate in? And how might we characterize such forms of participation? What kinds of mathematical arguments are both worth having and accessible to young children? Our research is aimed squarely at these concerns. Here we have selected one representative lesson among many that were part of a multiyear investigation to introduce space and geometry into the mathematics education of young children (Lehrer et al., 1998; Lehrer et al., 1999). Beginning with claims about “fat” and “skinny” forms, and with attendant methods of looking, the teacher of this second grade classroom helped children invent and coordinate forms of conceptual and procedural knowledge so that the ontological status of the claims changed. What was first tentative and a matter of everyday talk eventually became certain and a matter of mathematical talk, including a series of conjectures, refutations, and ultimately, informal proof. Along the way, children were guided to discover the concept and use of units of measure. This is an important mathematical milestone, from both disciplinary and developmental perspectives.

Our first analysis rendered the topology of the semantic structure of the argument as a directed graph. The graph affords clear visualization of what participants were talking about and how various senses of the mathematics of area measure—as conceived, as performed, and as historically rooted—were coordinated. Although mathematicians often refer to these multiple senses of practicing mathematics, the graph makes clear their emergence in the talk of this classroom. Indices of centrality, a measure of interconnection among particular mathematical ideas and operations, predicted what children recalled about the structure of the argument. In the reflective segment of the conversation, children who had not participated much (or at all) in the prior interaction proposed novel variants of procedures that were enacted earlier and traced the implications of these procedures (dealing with the use of units) for the concept of area. Their later recollection of the specific connections among concepts and procedures aligned well with the measures of centrality of these concepts that were obtained by analysis of the graph representing this earlier conversation. Core considerations in the argument, like unit, were instantiated in the graph of classroom discourse as an idea that could be easily reached from other ideas and procedures, once it was constructed.

Our graph-theoretical model for the structure of the argument is a novel methodology in the field of discourse analysis. Our explorations applying methods of graph theory as a means of analyzing classroom talk revealed (a) a means of visually summarizing the significant features (content and structure) of the entire argument, (b) several measures that provided a means of quantifying further description and summary of the structure of the lesson, and (c) in particular, the measure of centrality that proved to be predictive of student recall of the argument. This study explored only a small subset of the measures that could be extracted from the graph; the potential utility of other measures warrants further investigation. Based on these results, we believe that applied graph theory could be an extremely fruitful method for dealing with much larger chunks of discourse than currently-used methods in discourse analysis allow. In addition to opening up new methodological territory, the first analysis offers a global view of the structure and content of the argument, providing context for the local view provided by the second analysis.

The second form of analysis was aimed at understanding how the structure of the argument made visible by the graph was constructed. Here, Goffman's notions of participant frameworks proved fruitful for clarifying how the classroom teacher animated discussion and brought into contact the different mathematical senses displayed in the graph (e.g., different ideas about the nature of space covered, methods, prior experiences). By helping children articulate competing claims and by demanding that they voice the entailments of these claims, the teacher animated a variety of mathematical senses. She also communicated that precision and generalization are both valued in the discipline of mathematics. In this conversation, both precision and generalization functioned to resolve competing claims. Several of the students participated as if they were their teacher by listening to and revoicing the speech or actions of their classmates. These are clear indications of identity hinged to participation in the collective activity. The structure of the argument formed an arena for children's participation, even as the emotional markers of their teacher (e.g., her obvious delight in some of their mathematically fruitful thoughts and actions) and "I" statements of both teacher and children constituted an individualized linguistic "setting" for this activity (Lave, Murtaugh & De la Rocha, 1984). Thus, this second form of analysis shed light on the conversational moves by which participation was engendered.

In our view, both forms of analysis are necessary. One could envision teaching that encouraged children to participate, yet produced forms of participation not representative of mathematical reasoning. We suspect that students might enjoy such an environment but learn little of mathematical consequence. One could also envision well-structured mathematical arguments conducted in a relatively impoverished conversational space of initiate-respond-evaluate, with attendant negative consequences for identity and interest in mathematics. The two complementary analyses suggest that children in this classroom learned to participate in an interesting and ultimately fruitful mathematical argument.

We close by considering some of the potential relationships between everyday and mathematical talk that are suggested by our analysis. Clearly, the argument was sparked by children's interest in resolving contesting claims, something characteristic of any argument. Hence, the initial staging was commonplace, as were initial efforts to appeal to evidence of "looks like." The conversation ranged over a variety of artifacts, including paper strips folded and oriented in various ways, and the conversational turns were governed by event structures common to stories, like initiating events, and by conarration of explanatory sequences (e.g., why two rectangles might look different yet cover the same space). These, too, are commonplace, although perhaps all the more remarkable in light of their ubiquity in everyday thinking (Goodwin, 2000; Hall, 1999; Ochs, Taylor, Rudolph, & Smith, 1992; Wertsch, 1998). And we have already mentioned drawing upon children's sense of make believe or "suppose" as a congruence between everyday and mathematical thought. Thus, there were many parallels between the argument that children had about rectangles and other kinds of everyday thought that transpire in and out of school.

Although these everyday resources were at play, nonetheless, the mathematical argument here departs from everyday controversy. First, the argument's resolution involved the creation of a notation that rendered the concept of unit at once mobile and unambiguous. The production and use of notational systems is one hallmark of mathematical practice: "To do mathematics is to be involved in the corporeal practice of making physical inscriptions"

(Rotman, 1993, p. 33). Here children put the notation to use to resolve lingering ambiguities and at the end of the lesson, to embark on new investigation with these units as conceptual tools. Second, the scope of the argument was tilted toward the general, in both local statements (e.g., children's use of impersonal You to emphasize the distance between a method and any particular actor) and in the overall structure of the argument. Children eventually concluded that any set of figures might look different but have the same area measure. Third, and related to the first two points, the argument was ultimately resolved by finding ways to restructure the goals of the task to make definition of "cover" precise, so that methods and their conceptual implications spiraled toward rendering the conclusion certain, not just highly probable. Fixed in notation, pushed toward the general, and honed toward increasing precision—these truly mathematical aspects of children's argument emerged from the structure of the talk, but were fashioned by the moment-to-moment conversational exchanges orchestrated by the teacher. Our two complementary levels of analysis permit us to illustrate development as a historical process—one that includes both the interpretation of how constituent events came to be and a larger-scale view of how these events participate in an emergent structure.

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Appendix: Coding of classroom utterances for mathematical senses of area

Concepts

Concept codes classify the mathematical senses of area evident in the classroom conversation. Concepts are coded only when they are explicitly marked in the discourse, so that only those mathematical concepts that are openly discussed by the participants are coded, not those that the discourse and actions stimulate in the mind of a more-mathematically-literate coder.

(C1) Amount of space covered ("Taken up"). "Which of these quilts takes up more space: A, B, or C?"

"They cover up the same space."

(C2) Congruence/area distinction. Reference to how forms that look different might or might not be the "same." An appearance, "reality" conflict.

"How can they look so different and cover the same amount of space?"

"They look very different 'cause one is skinny and one is fat."

(C3) Additive congruence. Knowledge of the implication that congruence can be established by reallotment (partitioning and rearranging partitions).

S1: "So what does that mean?" (Referring to procedural sequence of cutting and rearranging.)

S2: "A & B are the same."

S1: "Same what?"

S3: "Same size."

(C4) Relationship between size of unit and count of unit. The number of units that will fill a particular shape is dependent on the size of the unit.

S1: How can Sam split that shape into 3 parts? I thought it was 4?

S2: No, it's a different way than Susan did.

S1: But that shape is 4!

S2: But Sam's using the long side, not the short side. That's why he only gets 3.

(C5) Relationship between count of units and amount of space covered. Two shapes have the same count of units if and only if they cover the same amount of space.

S1: So which shape, A, B, or C, covers the most space?

Students: All of them!

S2: they're the same, because they all have 12 squares.

Procedures

Procedure codes capture the physical actions carried out by the participants. There are three major classes of procedures (perception, partition, unit). The first class of procedures relies primarily on visual *perception.* Children compare the congruence of two or more objects by recourse to their appearance. A more sophisticated variation occurs when children believe that apparent differences in area along one dimension are compensated for by related differences along a second dimension.

(P1) Rudimentary comparison. (Two or more forms are contrasted by their appearance. Congruent areas look the same).

"Shape A is bigger because it's taller."

(P2) Compensation. Perceptual differences along one dimension are compensated for by differences along a second dimension.

"Shape A and Shape B might be the same size, it's hard to tell, because Shape A is longer and thinner, but shape B is shorter and fatter."

The second class of procedures *partition* space in a variety of ways to establish additive congruence. One set of procedures partitions space into parts, not all of which are congruent,

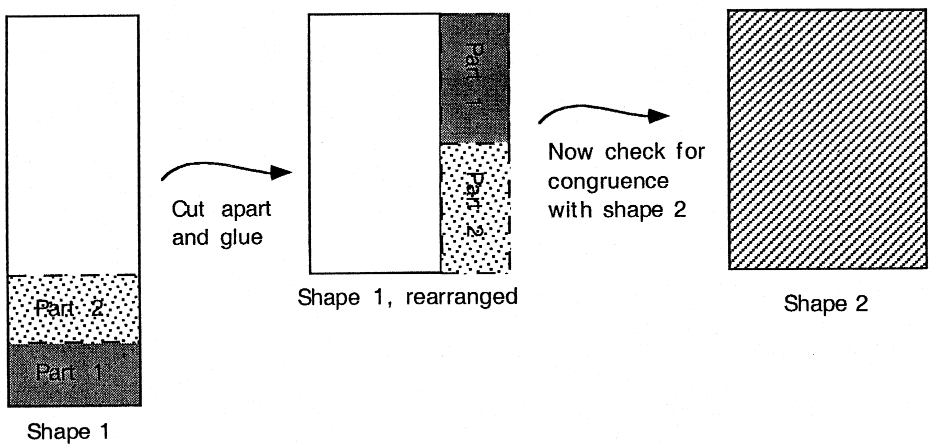

Fig. A1. Rearranging parts in Shape 1 to make it congruent with Shape 2.

and then rearranges these parts to establish congruence between two or more forms. The second set of procedures uses congruent partitions and their rearrangements to establish congruence.

(P3) Rearranging parts. By rearranging parts in one shape, it is made congruent with another. The parts are not all congruent.

"So I would cut this part [Part 1] off, turn it, and glue it up at the top over here, and then cut this part off [Part 2], turn it, and glue it down at the bottom over here" (accompanied by gestures to indicate actions represented in the pictures below. Fig. A1).

(P4) Matching parts. The congruence between two shapes is established by part-by-part correspondence, typically by overlaying each part (or describing or refering to such an action). Within a shape, the parts are not necessarily all congruent.

S1: We'd have to cut it in half [referring to the piece].

S2: OK. Imagine that that's exactly in half. And then what would we do with those pieces?

S1: We'd take this one and place it in kind of the top half of this, [points to top of C] and take the other one, and place it under it [points to bottom of C].

(P5) Traced part used as template. After tracing one part of one shape on the blackboard, this tracing is used as a template for establishing congruence of parts. (Fig. A2)

(P6) Form congruent partitions within shape. Dividing a shape into congruent parts (such as halves, thirds, fourths, etc.), usually by folding to establish congruence within the shape. A shape is folded and reopened to show all three thirds.

(P7) Use template to form congruent partition between shapes. Use one fractional part of one shape (i.e., one part from a congruent partition) as a template for dividing a second shape into a complete congruent partition.

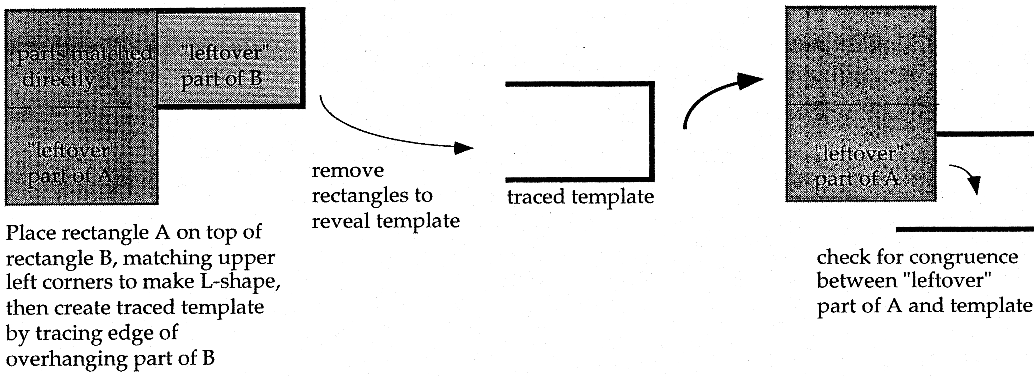

Fig. A2. Example of traced part used as template.

As in code P4, "matching parts," the action establishing congruence is applied to individual parts, but P7 adds the condition that the student uses the same part from shape one to match every fractional part in shape two. This contrasts to code P4, where each of the parts in shape one is matched with corresponding parts in shape two. (Fig. A3)

The third class of procedures *unitize and count* space to establish congruence. Congruent partitions are recruited as units of measure which are quantified, resulting in area measure as a count of congruent partitions that can be recruited to measure the area of any of the forms.

(P8) Counting parts in partitions. Counting the number of parts within a particular partition of a shape.

"One, two, three, . . . ten, eleven, twelve. [points to each square as he counts it]."

"Can you get four pieces this big out of that shape?"

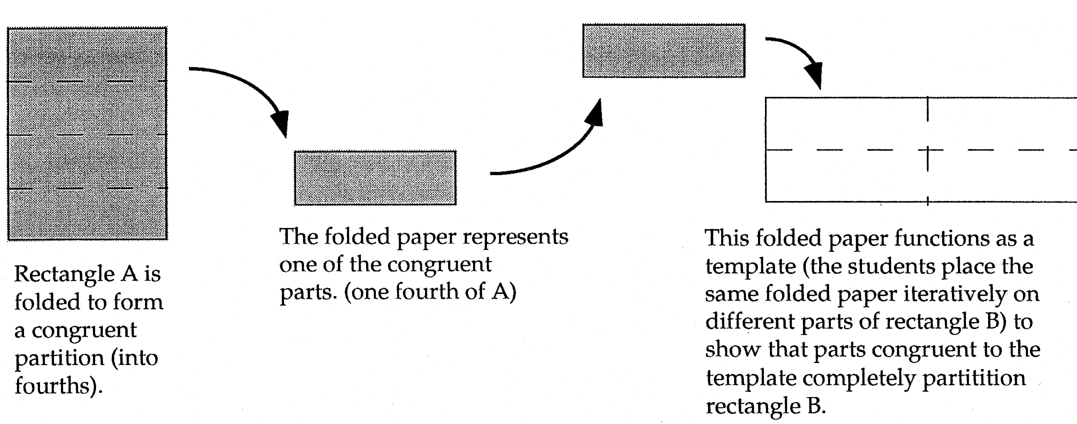

Fig. A3. Folding paper to represent congruence of Rectangles A and B.

(P9) Comparing partitions. Comparing the size of parts or number of parts in several different partitions of the same shape or across shapes. Partitions become objects of discussion and number of parts, size, or congruence of partitions are compared or distinguished.

"Sam needed to make three parts to cover it. How many parts would you have to make to cover it your way?"

(P10) Using count of units as measurement. Counting number of units as a quantification of amount of space covered.

"We found out that all of them had 12 core squares."

(P11) Reconfiguring templates as groups of units. Counting the number of units that cover one part of a congruent partition.

"There are 4 of these parts, and each one is 3 squares."

Or a student refers to the action of replacing congruent part (template) by a group of units, "Then she could make those parts into 3 core squares."

(P12) Counting groups of units (templates). Skip-counting units in groups, or using the count of units in one template plus knowledge of how the template partitions the shape, to determine how many units are in the whole shape.

"You could just make 3 rows of 4."

Or after establishing that a particular template can be covered by 3 units, counting the total number of units in the shape by pointing to each template and skip-counting by three "So she'd have three, six, nine, twelve."

(P13) Using unit as unit of construction. Using the unit of area measure as a component for constructing new shapes.

"I want you to pull twelve of those orange squares out of the file drawer, OK? And I want you to arrange them into Shape A. OK?"

Quilt history

(Q1) General reference to quilts. Mention of quilt history, but no explicit mention of core square, amount of cloth used, or any reference to how the same core squares can result in different looking quilts.

"I have quilts and I have part done and that's what you see here. . .I'm working on three quilts at once, but when I sewed my patches together, I have three very different looking shapes so far."

(Q2) Amount of cloth in quilt. Explicit reference to the amount of cloth used to create a quilt (as a means of grounding the concept of area).

"So which of these takes more cloth to make?"

(Q3) Reference to core squares. Mention of quilt history and explicit references to core squares.

"It's a core square!" or "There are 12 core squares."

(Q4) Different shapes can be constructed from the same core squares. Explicit reference to experiences using the same number of core squares to build quilts with different shapes (e.g., rectangles vs. squares vs. triangles) or different arrangements (e.g., a 2 × 2 block design vs. a 1 × 4 row of core squares).

Notation

This code marks instances when a white "core square" is used to represent a unit of area measure. The square is used to measure, embodies a concept of unit, and draws upon a previous history of inscription in quilt design.

(N1) Notation. The core square is used as a unit to measure the rectangles, or it is referred to as a means of performing this measurement. N1 is also applied whenever the measurement of the rectangle is stated as some number of these core squares.

References

Anderson R. C., Chinn C., Chang J., Waggoner M., & Yi H. (1997). On the logical integrity of children's arguments. *Cognition and Instruction, 15,* 135–167.

Arcavi A., Meira L., Smith J. P., & Kessel C. (1991). Teaching mathematical problem solving: an analysis of an emergent classroom community. In A. H. Schoenfeld J. Kaput E. Dubinsky (Eds.) *Research in collegiate mathematics education, Vol. III* (pp. 1–70). Providence, RI: American Mathematical Society.

Ball, D. L., & Bass, H. (2000). Making believe: the collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, Constructivism in education*. (pp. 193–224). Chicago: University of Chicago Press.

Brown A. L. (1992). Design experiments: theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences, 2,* 141–178.

Cobb, P. (2001). Supporting the improvement of learning and teaching in social and institutional context. In S. Carver & D. Klahr (Eds.), *Cognition and instruction: 25 years of progress* (pp. 455–478). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb P., Wood T., & Yackel E. (1993). Discourse, mathematical thinking, and classroom practice. In E. A. Forman N. Minick C. Addison Stone (Eds.) *Contexts for learning* (pp. 91–119). Oxford: Oxford University Press.

Davis P. J., & Hersch R. (1981). *The mathematical experience.* Boston: Houghton Mifflin.

Forman E. A., Larreamendy-Joerns J., Stein M. K., & Brown C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. *Learning and Instruction, 8,* (6):527–548.

Goffman E. (1974). *Frame analysis.* Cambridge, MA: Harvard University Press.

Goffman E. (1981). *Forms of talk.* Philadelphia: University of Pennsylvania Press.

Goodwin C. (2000). Vision and inscription in practice. *Mind, Culture, and Activity, 7* (1 & 2):19–36.

Hall R. (1999). The organization and development of discursive practices for "having a theory." *Discourse Processes, 27,* 187–218.

Harel G., & Sowder L. (1998). Students' proof schemas: Results from exploratory studies. In A. H. Schoenfeld J. Kaput E. Dubinsky (Eds.) *Research in collegiate mathematics education, III* (pp. 234–283). Washington, DC: Mathematical Association of America.

Healy L., & Hoyles C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31,* 396–428.

Kline M. (1980). *The loss of certainty.* Oxford: Oxford University Press.

Krummheuer G. (1995). The ethnography of argumentation. In P. Cobb H. Bauersfeld (Eds.) *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.

Kuhn D. (1989). Children and adults as intuitive scientists. *Psychological Review, 96,* 674–689.

Lakatos I. (1976). *Proofs and refutations.* Cambridge: Cambridge University Press.

Lampert M. (1990). When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. *American Educational Research Journal, 27,* 29–63.

Latour B. (1990). Drawing things together. In M. Lynch S. Woolgar (Eds.) *Representation in scientific practice* (pp. 19–68). Cambridge, MA: MIT Press.

Lave J., Murtaugh M., & de la Rocha O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff J. Lave (Eds.) *Everyday cognition: Development in social context* (pp. 67–94). Cambridge, MA: Harvard University Press.

Lehrer R., Jenkins M., & Osana H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer D. Chazan (Eds.) *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Mahwah, NJ: Erlbaum.

Lehrer R., Jacobson C., Thoyre G., Kemeny V., Strom D., Horvath J., Gance S., & Koehler M. (1998). Developing understanding of space and geometry in the primary grades. In R. Lehrer D. Chazan (Eds.) *Designing learning environments for developing understanding of geometry and space* (pp. 169–200). Mahwah, NJ: Erlbaum.

Lehrer R., Jacobson C., Kemeny V., & Strom D. (1999). Building on children's intuitions to develop mathematical understanding of space. In E. Fennema T. A. Romberg (Eds.) *Mathematics classrooms that promote understanding* (pp. 63–87). Mahwah, NJ: Lawrence Erlbaum Associates.

Martin W. G., & Harel G. (1989). Proof schemes of preservice elementary teachers. *Journal for Research in Mathematics Education, 20,* 41–51.

Mayr E. (1997). *This is biology.* Cambridge, MA: Harvard University Press.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Newell A., & Simon H. A. (1972). *Human problem solving.* Englewood Cliffs, NJ: Prentice-Hall.

Ochs E., Taylor C., Rudolph D., & Smith R. (1992). Storytelling as a theory-building activity. *Discourse Processes, 15,* 37–72.

O'Connor M. C., & Michaels S. (1993). Aligning academic task and participation status through revoicing: analysis of a classroom discourse strategy. *Anthropology and Education Quarterly, 24,* 318–335.

O'Connor M. C., & Michaels S. (1996). Shifting participant frameworks: orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63–103). Cambridge: Cambridge University Press.

Outhred L. N., & Mitchelmore M. C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education, 31*, 144–167.

Piaget J., & Inhelder B. (1956). *The child's conception of space.* London: Routledge and Kegan Paul.

Piaget J., Inhelder B., & Szeminska A. (1960). *The child's conception of geometry.* New York: Basic Books.

Rittle-Johnson B., & Siegler R. (1998). The relation between conceptual and procedural knowledge in learning mathematics: a review. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 75–110). East Sussex, UK: Psychology Press, Taylor & Francis.

Rogoff B. (1990). *Apprenticeship in thinking: cognitive development in social context.* New York: Oxford University Press.

Rotman B. (1993). *Ad infinitum.* Stanford, CA: Stanford University Press.

Rotman B. (1988). Toward a semiotics of mathematics. *Semiotica, 72,* 1–35.

Schauble L., & Glaser R. (1990). Scientific thinking in children and adults. In D. Kuhn (Ed.), *Developmental perspectives in teaching and learning thinking skills*: *Contributions to human development*, Vol. 21; (pp. 9–27). Basel: Karger.

Schoenfeld A. H. (1986). On having and using geometric knowledge. In J Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 225–264). Mahwah, NJ: Lawrence Erlbaum Associates.

Schoenfeld A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education, 20,* 338–355.

Silver E. A. (1986). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 181–198). Mahwah, NJ: Lawrence Erlbaum Associates.

Simon H. A. (1969). *The sciences of the artificial.* Cambridge, MA: MIT Press.

Tharp R. G., & Gallimore R. (1988). *Rousing minds to life.* Cambridge: Cambridge University Press.

Wenger E. (1998). *Communities of practice: Learning, meaning and identity.* Cambridge: Cambridge University Press.

Wertsch J. (1998). *Mind as action.* Oxford: Oxford University Press.

Yackel E., & Cobb P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27* (4):458–477.