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Resources for Refining Mathematical Conceptions: Case Studies in Learning About Linear Functions

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In this article, I describe the refinement of a conception in the domain of linear functions, the use of the x-intercept in equations of the form $y = mx + b$, to illustrate the resources used by students to refine a conception. The examples presented show how students refined their use of the x-intercept by (a) narrowing the problem contexts in which they applied this conception, (b) making connections between conceptions, (c) using a mathematical procedure, and (d) refining their verbal descriptions of how the graph of a line changes as a result of a change in the equation. The analysis draws on diSessa's (1993) theory of science learning and the recommendations made by Smith, diSessa, and Roschelle (1993) for describing conceptual change. The results extend this prior work from science to mathematics and identify resources used by students that teachers can draw on to support the learning of mathematical concepts.

Learners often develop conceptions—coherent, stable, and robust ideas—that are different than expert conceptions in a domain. These "misconceptions" can sometimes interfere with students' learning of mathematical concepts. To support learners in refining their conceptions and developing more expert conceptions, it is not sufficient to merely know that "misconceptions" exist: We need to understand the process of conceptual change that enables learners to transform and refine their conceptions to more closely fit with the desired understandings for a domain. An important aspect of understanding conceptual change is identifying the positive resources that learners already have and use in the process of refining conceptions

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(Smith, diSessa, and Roschelle, 1993). Although there have been many studies of misconceptions in mathematics learning, few have identified such resources.

One way to identify student resources for conceptual change is to study how student conceptions change in supportive contexts. In this article, I investigate the resources students used while learning about linear functions in a particularly powerful context using graphing software and discussing their work with another student. I present examples from a study in which 18 ninth-grade students participated in videotaped discussion sessions with a peer. These students had been previously observed during classroom lessons about functions and completed written assessments before and after the discussion sessions. During the discussion sessions students worked in pairs after school, using graphing software to explore the connections between equations of the form $y = mx + b$ and their graphs. Interestingly, the data show that even though students generated several common misconceptions, many students refined their initial conceptions. What resources enabled these students to learn successfully and refine their initial conceptions?

A videotape analysis of six case studies seeking to describe how students refined their initial conceptions and to identify the resources that enabled conceptual change provides an answer to this question. Students were learning about the relationship between the graphs of lines and their equations. During this learning process, many of the students initially used the x-intercept in the form $y = mx + b,$ either as the m or the b in an equation, or to describe the effect of changing b on a line. In this form of the equation the two parameters needed to determine the equation of a line are the slope, $m,$ defined as the ratio of the change in y to the change in x between any two points on the line, and the y-intercept, $b,$ the y-coordinate of the point where the line crosses the y-axis. Although the x-intercept—the x-coordinate of the point where a line crosses the x-axis on the graph—may be perceptually salient, it is not directly accessible in this form of the equation for a line.[1]

Using the x-intercept to generate the parameters in this form of the equation is not always wrong. For example, when the slope is 1, $-b$ is the x-intercept. Also, changing b in an equation does, in effect, move lines along the x-axis (as well as along the y-axis), so that descriptions of the effect that changing b in an equation has on a line that include horizontal translation are not necessarily wrong. However, experts usually focus on movement along the y-axis as result of changing b. This choice reflects the fact that lines move exactly b units up or down the y-axis when, for example, an equation is changed from $y = x$ to $y = x + b$. Relating the parameter b in an equation to the movement of a line along the x-axis is a more complicated relationship because lines move either $-b$ units along the x-axis—in the case of m

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[1]This form of the equation can be algebraically transformed so that the x-intercept is more directly accessible. For example, if the equation is transformed into $y/m - b/m = x,$ $-b/m$ is the x-intercept and appears as a single term in the equation. If the equation is changed into the form $x/a + y/b = 1,$ then a is the x-coordinate of the x-intercept and can be read directly from this form.

$= 1$—or $(-b)/m$ units along the x-axis—in the case of $m \neq 1$. Although focusing on movement along the x-axis is not wrong, focusing on movement along the y-axis is the simplest possible correlation between the two representations and is thus not an arbitrary choice.

Although this "misconception" can be difficult to overcome, some of these students successfully developed a better understanding of the connections between an equation and its graph and of the appropriate role for the x-intercept. The analyses of the discussions revealed four types of resources for conceptual refinement: (a) specific cases, (b) connections to other conceptions, (c) mathematical procedures, and (d) descriptive language. In this article, three case studies are presented to illustrate these resources, which are proposed as part of a general repertoire of resources available to learners for conceptual refinement.

THEORETICAL FRAMEWORK

The analysis of conceptual change presented here is based on diSessa's (1993) theory of science learning in several ways. First, the general perspective of knowledge systems as "knowledge in pieces" provides a basis for this analysis of student conceptions. The use of the x-intercept is assumed to be one of many pieces of student knowledge in this domain, which is connected (or not) to other pieces of knowledge. Second, the study documents how knowledge pieces are invoked and refined according to application context, another important aspect of diSessa's theory. And, lastly, the study provides examples of how the refinement of an intuitive idea occurs by narrowing the contexts in which it is applied and making connections to other conceptions, as suggested in Smith et al. (1993).

The perspective on student conceptions used in this study also addresses the recommendations made by Smith et al. (1993) for analyzing student conceptions and describing conceptual change. They recommended analyses that address the continuity and similarities between novices and experts and that see student conceptions as functional and valid: "Pervasive ideas arise from their usefulness or from their connection to useful ideas. This is an excellent heuristic to help discover the origin of naive ideas. It also undermines the idea that uninstructed learning is almost always flawed" (p. 59).

Rather than seeing student conceptions merely as right or wrong, regardless of context, they suggested that conceptions should be understood in terms of their productivity and characterized in terms of the contexts in which students use these conceptions: "Misconceptions, if properly understood will generally turn out to be the result of extending an adequate conception to a context in which it is inadequate. Productive or unproductive is a more appropriate criterion than right or wrong" (p. 59).

They also recommended analyses that characterize conceptions in terms of how they are connected to the students' larger knowledge framework, how one concep-

tion is related to others, and how a conception changes gradually. Following this framework, two of the four aspects of conceptual change described here involve application contexts and connections to other conceptions.

Although diSessa's (1993) theory provided a framework for understanding students' conceptions, it has not addressed the role of language in conceptual change. In this article, I extend diSessa's theory by including the refinement of verbal descriptions as an important aspect of conceptual refinement. Conceptual change will be assumed to be a constructive process that involves a dialectical relation between individual and social aspects of knowing and learning, including language (Walkerdine & Sinha, 1978). Another analysis of this data focused on how the peer discussions supported student's constructions of shared descriptions (Moschkovich, 1996). Although the article does not include a discussion of the relation between conceptions and language, I assume that knowledge construction is mediated by symbol systems including language (Lucy & Wertsch, 1987; Vygotsky, 1974, 1978, 1987).

Following Smith et al. (1993), I define a *conception* as an idea that is stable over time, the result of a constructive process, connected to other aspects of a learner's knowledge system, and robust when confronted with other conceptions. In the case studies described, the x-intercept was an important student conception in the domain of linear functions.

PRIOR RESEARCH ON CONCEPTIONS OF LINEAR FUNCTIONS

Research on student conceptions has documented specific student conceptions, described how they are at variance with expert ideas, and suggested instructional strategies for addressing particular misconceptions. Research in mathematics learning has provided detailed accounts of student errors and misconceptions in several areas—whole number subtraction (Brown & Burton, 1978; Brown & Van Lehn, 1980; Burton, 1982; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1982), rational numbers (Post, Wachsmuth, Lesh, & Arcavi, 1985), algebra (Clement, 1980, 1982; Matz, 1982), graphs (Bell & Janvier, 1981), and functions (Herscovics, 1989; Moschkovich, 1992; Moschkovich, Schoenfeld, & Arcavi, 1993; Schoenfeld, Smith, & Arcavi; 1994; for a review, see Leinhardt, Zaslavsky, & Stein, 1990). From this research we know that conceptual competence in the domain of linear functions includes much more than knowing procedures. It involves understanding the connections between the graphical and algebraic representations, knowing which objects are relevant in each representation, and knowing which objects are dependent or independent.

One student conception documented in this domain (Goldenberg, 1988; Moschkovich, 1989, 1990, 1992; Schoenfeld et al., 1994) is that the x-intercept is

relevant for equations of the form $y = mx + b$, that is, it should appear in the equation, either in the place of m or in the place of b. In a case study Schoenfeld et al. referred to this conception as a "three slot schema" for the form $y = mx + b$, describing a student who searched for three pieces—the slope, the y-intercept, and the x-intercept—to generate equations of this form. Goldenberg noted that some students described lines as moving to the right or to the left as a result of changing the b in an equation. Students have also been observed using this conception in classroom settings (Moschkovich, 1990, 1992).

In the remainder of this article, I will focus on the x-intercept conception and how it changed over time. The case studies presented in this article are part of a larger study, which was designed to explore student conceptions and learning in the domain of linear functions. The results of this study are first summarized to set the context for the case studies. Interested readers may find the full results in Moschkovich (1992). The case studies presented following the summary focus on the analysis of the discussions for six pairs and the resources that students used to refine this conception.

SUMMARY OF STUDENT'S USE OF THE *X*-INTERCEPT

The students who volunteered to participate in the study were from an exemplary pilot first-year algebra course. Classroom lessons were observed during two chapters covering functions (see Table 1). Following these two chapters, 18 students completed written assessments and participated in videotaped discussion sessions with a peer of their choice, using Superplot (Steketee, 1985)—a graphing utility that allows students to graph equations—to explore linear functions. In this section of the article, I briefly summarize the results of the classroom observations and written assessments relevant to students' use of the x-intercept.

The students were from two classrooms observed earlier in the school year (Moschkovich, 1990) during two chapters on functions, which included (a) modeling of real-world situations, (b) use of graphing calculators and computer software, and (c) student group work with some whole-class discussions. The curriculum was designed to include exploration and discovery, focus on mathematics as a process rather than on results or answers, support work in groups, and encourage students to discuss their ideas. These students attend an urban school that has about a 90% minority population of working class and lower middle class families. The students in this course were mostly 9th graders, although a few were 10th graders. They were neither honors nor remedial students, and the classes were heterogeneous in terms of previous math achievement scores.

The assessment and discussion problems addressed the student conceptions documented in previous research in this domain (Goldenberg, 1988; Schoenfeld et al., 1994) and two conceptions noted during the classroom observations (Moschk-

TABLE 1
Data Sources

	Duration	Data
Classroom observations	Observed two 6-week chapters, the first covering linear functions (Fall), the second covering quadratic functions (Spring), in two classrooms.	Field notes, audiotapes of two groups in each classroom. Videotapes of two lessons on the computer for one pair in each classroom.
Written assessments	Approximately 1 hr. Completed in the Spring after the second chapter, the pretest immediately preceding the discussion sessions and the posttest 2 days after completing the discussion sessions.	18 students completed 28 questions.
Peer discussion sessions	There were no time constraints. Sessions lasted from 2 to 4 hr over a period of at least 2 days and at most 4 days. Sessions were conducted in a classroom after school.	Videotapes of all sessions.

ovich, 1990), using the x-intercept and conflating the effect of changing m and b. The responses on the pretest and posttest were used to document the extent of particular student conceptions and to assess any improvement after the discussion sessions. The videotapes of the peer discussions were analyzed to document the existence and nature of student conceptions, to explore changes in these conceptions, and to examine students' use of language.

Data collected in three settings—classroom lessons, written assessments, and the discussion sessions—show that the use of the x-intercept was widespread, stable, and robust. Students in the two classrooms were observed using the x-intercept to generate b and m either when generating an equation to produce a line or when generating an equation for a graphed line (Moschkovich, 1990), even though classroom instruction had focused solely on using the y-intercept to generate b or translate lines. The data from the written assessments and discussion sessions completed by 18 students show that students repeatedly used the x-intercept, despite the fact that the form of the equation students were using during the study—and had used most frequently during the classroom lessons—$y = mx + b$, does not employ or address the x-intercept. The results of the study show that the use of the x-intercept was a coherent, stable, robust, and sometimes productive conception, rather than a superficial error (Moschkovich, 1992).

There were three uses of the x-intercept documented: (a) as the b in an equation, (b) as the m in an equation, and (c) to describe the effect that a change in an equation

has on a line. These uses of the x-intercept are dependent on the problem context.[2] In particular, the value of the slope is relevant to whether using the x-intercept for b in an equation is applicable—the opposite of the value of the x-coordinate of the x-intercept does indeed produce the correct value of b but only when $m = 1$. A second problem context where the use of the x-intercept can be productive is when the x-intercept is explored as a reflection of changes in the slope (rather than used as m itself).

Although the study documented three different ways in which students invoked the x-intercept when using the form $y = mx + b$, I consider the use of the x-intercept as a unitary conception underlying these different student responses. Specific student responses or errors, such as using the x-intercept for b in an equation, are considered to be a reflection of the underlying conception that "the x-intercept is relevant," rather than the conception itself (Confrey, 1990; Matz, 1982). Rather than considering each use as a separate conception, the x-intercept was analyzed as a unitary conception that is manifested differently depending on the problem context.

The term x-*intercept* can refer to either the point where a line crosses the x-axis or to the number that is the x-coordinate of this point (and which can be used in an equation). During the following discussion, I will continue to use the terms y-*intercept,* x-intercept, and *slope* sometimes loosely as general terms for the objects in either representation. However, for the sake of clarity, I will also sometimes distinguish between the graphical and the algebraic objects. In the case of slope, the graphical object is the slant of a line and the algebraic object is the number (or letter) appearing in an equation; in the case of the intercepts the graphical objects are points on a line and the algebraic objects are the number (or letter) appearing in an equation (for the y-intercept), or the result of an algebraic manipulation of the form $y = mx + b$ (for the x-intercept). I will use the following notation to distinguish between graphical and algebraic objects:

b_E: the algebraic y-intercept b_G: the graphical y-intercept
m_E: the algebraic slope m_G: the graphical slope
a_E: the algebraic x-intercept a_G: the graphical x-intercept

The m and b correspond to the parameters in the form $y = mx + b$; a corresponds to the parameter a in the equation $x/a + y/b = 1$, where a is the x-coordinate of the x-intercept, and b is the y-coordinate of the y-intercept. The subscript "E" stands for equation and the subscript "G" stands for graph.

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[2]I use the phrase *problem context* to differentiate between the larger social context and the particular problem a student is working on. The problem context includes the details of an equation, such as the value of the slope, and also how students define, interpret, and generate problems (e.g., whether they are comparing two lines or exploring the slope).

Students' uses of the x-intercept could be considered as a simple error or a mismatch with the fact that the x-intercept simply does not appear in equations of this form. On the other hand, these uses of the x-intercept can be seen as a conception that is reasonable, reflects the mathematical complexity of this domain, is applicable in some contexts, and can be refined. The explanations students provided on the written assessments show that the use of the x-intercept was not merely a superficial mistake or a careless switching of x and y. The results from the written assessments show that this conception was robust because this conception did not disappear completely even after the discussion sessions.[3]

Responses on any of the 28 pretest questions showed that 13 out of 18 students (72%) used the x-intercept at least once—for either b or m in an equation or to describe lines as moving along the x-axis as a result of changing b in an equation and 10 students (55%) used the x-intercept two or more times. On the posttest, 11 students (60%) used the x-intercept at least once and 7 students (38%) used the x-intercept twice or more. On the pretest responses on any of the 28 questions, 13 students (72%) used the x-intercept at least once and 10 students (55%) used it two or more times. Comparing the pretest and posttest responses on any of the 28 questions,[4] the number of students describing lines as moving along the x-axis as a result of changing b in an equation at least once decreased from 6 to 2 (33%–10%), the number of students using the x-intercept for b at least once decreased from 12 to 11 (66%–60%), the number of students using the x-intercept for b twice or more increased from 5 to 7 (27%–38%), the number of students using the x-intercept for m at least once decreased from 5 to 1 (27%–0.5%), the number of students using the x-intercept for m twice or more decreased from 2 to 0 (10% to 0).

The data from the discussion sessions also show that the use of the x-intercept was more than a superficial mistake from which students could easily recuperate. This conception became a central point of the students' discussions. Most of the students spent a considerable amount of time discussing the x-intercept and making sense of their responses regarding the x-intercept. The results show that this conception can be refinded, because many students did improve on their use of the x-intercept on the posttest, this conception can be refined. In the rest of the article, I will focus on analyses that examined how students refined this conception and what resources they used.

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[3]The sequence of the assessment and discussion problems may have supported the over generalization that $b_E = -a_G$, because many of the initial problems had equations where $m = 1$. However, because students were documented using the x-intercept in the classroom as well, despite the strong classroom focus on the y-intercept, the sequencing of the problems could not have been the main source for this conception.

[4]There were two questions on the pretest that specifically addressed the use of the x-intercept. Improvement on these particular questions was highly statistically significant. Students' explanations of their answers to these two questions especially show that this was not superficial mistake (for a more

TABLE 2
Pretest and Posttest Scores for Case Studies

Case Study	Name	Pretest[a]	Posttest[a]
1	Mathew	1	29
	Jack	1	23
2	Mitch	22	30
	David	8	29
3	Monica	8	8
	Denise	7	17
4	Fred	29	31
	Harold	23	31
5	Marcela	12	25
	Giselda	2	13
6	Luis	1	16
	Laura	3	25

[a]Total 31 points.

OVERVIEW OF CASE STUDIES

To describe how students refined this conception during the discussions and the resources they used for conceptual change, six case studies were analyzed from the study previously described. In the rest of this article, I will focus on the analysis of the discussion sessions for these 6 pairs of students. These 6 pairs were chosen from the 9 pairs who participated in the study.[5] The analysis shows the transformation of this conception for these 12 students through their answers on the assessments and the discussion session data.

Assessment Results for Case Studies

As the scores in Table 2 show, there was an overall improvement for these 6 pairs, except for one student, Monica. The improvement on the assessments scores reflects in part some improvement in the use of the x-intercept. Table 3 shows the different uses of the x-intercept for each student in these six case studies. For the students in Case Study 1, Mathew showed no evidence of using the x-intercept on the posttest, whereas Jack's use remained the same, once for b_E. For the students in Case Study 2, Mitch showed no evidence of using the x-intercept on the posttest, whereas David

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[5]Of the original 9 pairs, 1 pair was dropped from the protocol analysis because their discussions were very limited and one of the students worked primarily on his own, rather than attempting to agree with his partner before recording their final answer. A 2nd pair was dropped because neither student showed improvement on the posttest. A 3rd dropped pair's discussion session was analyzed but used mainly to corroborate the analysis of the six case studies, and thus will not be discussed in any detail.

used it once for b_E. For Case Study 4, even though there was no evidence of the use of the x-intercept in either student's pretest or posttest responses these two student did invoke the x-intercept during their discussion sessions. For 3 of the students in Case Studies 3 and 5, the uses of the x-intercept on the pretest and posttest are more difficult to interpret. Marcela's use of the x-intercept for b_E decreased from 6 instances to 1, and her use of the x-intercept to describe line movement was not evident on the posttest showing a clear improvement. However, for Monica and Denise (Case Study 3) and Giselda (Case Study 5) their uses of the x-intercept on the posttest either remained the same, increased, or appeared in different categories.

Discussion Sessions

In the discussion sessions, the students explored slope and intercept using Superplot (Steketee, 1985)—a graphing utility that allows students to graph equations—and problems designed by the researcher on the basis of previous research (Goldenberg, 1988; Schoenfeld et al., 1994) and the classroom observations (Moschkovich, 1990). The following example illustrates the basic format used for the discussion problems.

In the first part of each problem (such as Problem 3a shown in Figure 1) students were given the equation $y = x$ and its graph and were asked to predict what changing the equation from $y = x$ to a target equation (in this case $y = x + 5$) would do to the line. In the second part of each problem they were given the graph of $y = x$ (or $y = 2x$) and a second line and were asked to predict what change in the equation $y = x$

TABLE 3
Pretest and Posttest Uses of the X-Intercept for Case Studies

Case Study	Name	Pretest			Posttest		
		G	b	m	G	b	m
1	Mathew	3	3	1			
	Jack		1			1	
2	Mitch	4	5				
	David		1	4		1	
3	Monica			1	1	2	
	Denise		1	1		2	
4	Fred						
	Harold						
5	Marcela	1	6			1	
	Giselda		1	2		3	1
6	Luis						
	Laura		2				

Note. G = graph; b and m correspond to the parameters in the form of $y = m + b$.

3a. If you start with the equation y=x
then change it to the equation y=x+5,
what would that do to the graph?
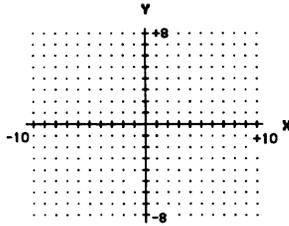
FIGURE 1   Sample discussion problem.

(or $y = 2x$) would generate the target graphed line. In some problems, Choice C read: "The line would flip to the other side of the y-axis" to address the effect of a negative slope. The problems discussed in the transcripts follow a format similar to the problem in Figure 1, except that Problems 1a, 1b, and 1c target the use of the x-intercept.

The introduction to the discussion sessions included a review of basic coordinate graphing skills, an explanation of how the computer graphs equations using a table of values generated by the students, a description of the guidelines for discussing the problem and recording their responses, and an explanation of key words used in the problems (*steep, steeper, less steep, origin, move up or down on the y-axis,*

tions for each problem before graphing, then to graph an equation, and then to choose answers and explanations again.

To structure the dialogue and promote the discussion of different conjectures, the students followed an instructional sequence similar to the Itakura method for classroom discussions in science (Hatano, 1988; Inagaki, 1981; Inagaki & Hatano, 1977). Students were asked to discuss and choose between several alternative predictions before graphing and record their choice on paper, including an explanation. Students were then asked to graph an equation to test their prediction and choose an agreed on answer and explanation once again after graphing. Students were told that they did not have to agree on their choices before graphing, and that their individual choices would be recorded on the paper and on the videotape, but that they had to agree on their choices after graphing. The conversations that ensued are labeled *discussions* because they meet the criteria for a mathematical discussion as "purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions" (Pirie, 1991, p. 143).

Case Study Analyses

Four analyses of the discussions for the case studies were conducted. Analysis 1 focused on documenting student conceptions by coding for occurrences of specific conceptions. This initial analysis showed that the use of the x-intercept, either in the place of m or b in equations was a common conception.[6] Analysis 2 explored how students' different uses of the x-intercept had changed throughout the discussion sessions. The transcripts for these six case studies were coded for changes in students' uses of the x-intercept for b_E over the duration of the discussion sessions.[7] Students' use of the x-intercept for b_E was coded as follows:

$a_G \leftrightarrow b_E$: An instance of a student using the x-coordinate of the x-intercept of a given line to generate b_E for an equation or an instance of a student using b_E in a given equation to generate the x-coordinate of the x-intercept of a line.

Opposite $a_G \leftrightarrow b_E$: An instance of a student using the opposite of the x-coordinate of the x-intercept of a given line to generate b_E for an equation

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[6]Another common student conception was the conflating of m and b (Moschkovich, 1990, 1996).

[7]Four students' refined their initial use of the x-intercept for m_E through exploring the relation between the x-intercept and m_E. However, other students did not refine this use of the x-intercept so explicitly in their discussion sessions. The use of the x-intercept for m_E was not analyzed because it did not follow a clear progression between different uses, as was the case of the x-intercept for b_E.

or an instance of a student using the opposite of b_E in a given equation to generate the x-intercept of a line.

$b_G \leftrightarrow b_E$: An instance of a student using the y-intercept of a given line to generate b_E for the equation or an instance of a student using b_E in a given equation to generate the y-intercept of a line.

The results of this coding were similar for the students in Case Studies 1, 2, 3, and 4. The results for these 8 students show that they refined their use of the x-intercept for b_E from using the x-intercept for b_E, to using the opposite of the x-coordinate of the x-intercept for b_E, and finally to using the y-intercept for b_E and no longer focusing on the x-intercept. By using the opposite of the x-intercept for b_E only when $m = 1$ these students narrowed the contexts in which they applied this conception.

Analysis 3 of the videotapes coded students' descriptions of lines. For three of the six case studies, I noted each instance where a student described a line, how a line had moved on the screen as a result of changing b or m, or compared two lines, focusing especially on the instances where students explicitly discussed their descriptions of what was on the computer screen. Case Studies 3, 4, and 5 were chosen for this analysis because these students explicitly discussed their descriptions. These students had initially described lines as moving "left" or "right" as well as "up" and "down" as a result of changing b_E. They later came to focus on the result of changing b_E as specifically moving lines up or down the y-axis. Their later descriptions also omitted mention of movement along the x-axis, reflecting not only an increasing coordination between the algebraic and graphical representations, but also a focus on the specific form of the algebraic representation used in the problems, $y = mx + b$.

Analysis 4 traced the changes in students' descriptions throughout the duration of the discussion sessions for Case Studies 3, 4, and 5. The ways in which these students' use of descriptive language changed was analyzed by comparing their descriptions of lines during two problems at the beginning of the sessions, during several problems in the middle, and during the last two problems in the discussion sessions.

RESOURCES FOR REFINING THE *X*-INTERCEPT

These analyses documented four types of resources for conceptual refinement:

1. *Specific cases:* One resource that students used for refining the x-intercept for b_E was specific cases. By working on different cases students narrowed the contexts in which they applied this conception and many stopped using it in contexts in which it was not applicable.

2. *Connections between conceptions:* Several students attempted or began to make a connection between the x-intercept and the slope as a resource for refining their use of the x-intercept for m_E.

3. *Mathematical procedures:* Two students used a procedure to generate the slope and recuperate from using the x-intercept for m_E.

4. *Descriptive language:* Students negotiated and refined their descriptions of the effect that changing b_E has on a line.

Excerpts from the discussion sessions for Case Studies 1, 2, and 3 will be used to illustrate in more detail the resources students used to refine this conception. Case Study 1 is an example of how students explored the specific cases in which the x-intercept was applicable and those in which it was not and attempted to make a connection between the x-intercept and the slope. Case Study 2 shows how two students used a procedure to generate m_E and recuperate from using the x-intercept as the slope. Case Study 3 shows how one student refined her use of descriptive language to focus primarily on vertical translation in her descriptions of the effect of changing b_E. The transcript materials are divided by excerpts for each problem. Each student turn is numbered consecutively within a problem excerpt. Students' gestures, the referent of a pronoun, or clarifications are included within brackets.

CASE STUDY 1: USING CASES TO NARROW APPLICATION CONTEXTS AND CONNECTING TWO CONCEPTIONS

Although these two students scored very low on the pretest (1/31 and 1/31), they showed substantial improvement on the posttest, both in terms of their overall scores (Mathew: 29/31, Jack: 23/31) and, in the case of Mathew, in terms of his use of the x-intercept. Although on the pretest Mathew used the x-intercept (in three instances for b_E, in one instance for m_E, and in three instances to describe line movement) there was no evidence of any use of the x-intercept in his posttest responses.

This pair of students used two resources to refine their use of the x-intercept, specific cases and another conception. They used specific cases to narrow the problem contexts in which they applied the x-intercept and they attempted to make a connection between the x-intercept and the slope of a line. During the first discussion problem, when predicting what the graph of the equation $y = x + 4$ would look like (Figure 2, Problem 1a), Mathew and Jack provided an explanation of why they expected b_E to correspond to the x-coordinate of a_G.

Mathew and Jack first answered that the line $y = x + 4$ would cross at *(4, 0)* because "in the equation you add 4 to x."

1a. Graph the equation y=x+ 4 below
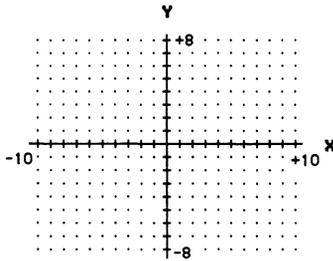
AFTER GRAPHING

A student said that this line would go through the axis at (4,0) because in the equation you add 4 to x.
Do you think this student was right?

| ____YES | ____NO |

Why or why not?

| ____YES | ____NO |

Why or why not?

Describe below how you could check your answer without the computer:

FIGURE 2   Problem 1a.

Excerpt 1: Mathew and Jack, Problem 1a

1. Mathew: "The student said the equation would go through the axis at (4, 0) [reading the question]." Yes, why? Because in the equation you add 4 to x ... now graph it? ... x plus 4 [Graphs the equation $y = x + 4$.]... what?
2. Jack: Oh, oh!
3. Mathew: I don't get this ... hmmm.
4. Jack: I forgot that it goes on this side [pointing to the II and III quadrants].

5.   Mathew: This is single x ... this is x [traces the line $y = x$ with his hand], right? Right through the middle is x ... so this should be x one, x two, x three, x four [successively pointing to lines through $(1, 0)$, $(2, 0)$, 3, 0), $(4, 0)$].

6.   Jack: What about negative 4? It might be over there [pointing to (4, 0)], just like this [points to their graph on worksheet].

7.   Mathew: This is the positive side [pointing to the positive side of the x-axis] ... that's negative 4 ... looks like that [referring to the line on the screen, $y = x + 4$] would be y equals ... x minus 4 ... hmmm.

As they graphed the equation $y = x + 4$, Mathew expected to generate lines of the form $y = mx + b$ by starting from the line $y = x$ and moving to the right along the x-axis (line 5), so that the equation $y = x + 1$ would cross the x-axis at $(1, 0)$, the equation $y = x + 2$ would cross the x-axis at $(2, 0)$, and so on. Mathew further specified that he expected the sign of b_E to be the same as the sign of a_G. After Jack suggested that the line for $y = x - 4$ might cross the x-axis (line 6) at $(4, 0)$, they graphed the equation $y = x - 4$. Mathew was then puzzled to see that a plus 4 in the equation would appear as an x-coordinate of minus 4 for a_G.

### Excerpt 1 Continued: Mathew and Jack, Problem 1a

11.   Mathew: Now this I don't get ... this I don't get at all.

12.   Jack: They're parallel ... see the only thing that's negative is this [pointing to the y-intercept of the line $y = x - 4$].

13.   Mathew: I know! ... This is x [points to origin], if it's taken away 4 why would it go this way (pointing to the point $(4, 0)$) and not that way [pointing to the point $(-4, 0)$]?

After graphing the equation $y = x - 4$, they concluded that b_E corresponds to the opposite of the x-coordinate of a_G, which is correct for cases where the slope is 1. They went on to use this conclusion in their first attempts during the next problem (Figure 3, Problem 1b) where the slope was 2.

Excerpt 2: Mathew and Jack, Problem 1b

1.   Mathew: What equation would this be? 3 ... x minus 3 ... cause that one was x minus 4.

2.   Jack: X negative 4 ... negative 3 ... You gotta turn it.

3.   Mathew: The angle, huh?

1b. Write an equation for the line graphed below.
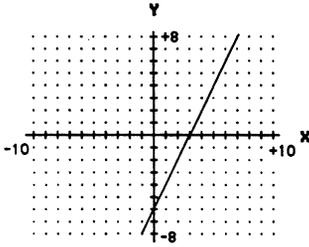
EQUATION:_____

Why?

AFTER GRAPHING

EQUATION:_____

Why?

A student said that the equation for this line was
$y=2x+3$ because the line goes through the x axis at $x=3$.
Do you think this student was right?

_____YES   _____NO

_____YES   _____NO

Why or why not?

Why or why not?

Describe below how you could check your answer
without the computer:

FIGURE 3   Problem 1b.

4.   Jack: That looks like a 5 or 6.

5.   Mathew: No, that would be more up so we want to say $5x$ minus 3? y ...
                $5x$ ... OK. We used $5x$ for the angle and -3 for the line position.
                OK, let's try it.

6.   Jack: [They graph the equation $y = 5x - 3$.] It's getting closer.

Mathew first proposed they use a_G as b_E. Jack then proposed they change the
slope of the line, later suggesting a slope of 5 or 6. Mathew then explained why the
slope is 5 and b_E is -3. They went on to discuss the slope and decided that the
number 5 produced and angle that was "too big" After graphing the equation $y =$

$3x - 3$, they continued discussing their choice for b_E. They first used the x-coordinate of a_G for b_E. After graphing the equation $y = 3x - 5$, they went on to discover that in the case of $m \neq 1$, b_E does not correspond to the opposite of the x-coordinate of a_G but to the y-coordinate of b_G:

Excerpt 2 Continued: Mathew and Jack, Problem 1b

30.	Mathew:	Not quite … it did it … It went down through six.
31.	Jack:	Steps of two [referring to the units in the graph on the computer screen] … this is one point five … it made it through six.
32.	Mathew:	Yeah, but this we gotta get to go up to three … So what do we try, two x?
33.	Jack:	All right. [They graph the equation $y = 2x - 6$.]
34.	Mathew:	Yeah, that did it. [Writes the answer for after graphing, $y = 2x - 6$, on the paper.]
35.	Jack:	But how does it work?
36.	Mathew:	How does it work?
37.	Jack:	Yeah … Cause I know this hit 6 [points to the y-intercept for the line $y = 2x - 6$].

Next Mathew and Jack went on to explore the relation between the x-intercept, a_G, and the slope, m_E, using trial and error to graph several equations, $y = -3x + 6$, $y = 6x - 3$, and $y = 3x + 6$, using either the x-coordinate of a_G, or its opposite, for m_E and for b_E.

Excerpt 2 Continued: Mathew and Jack, Problem 1b

44.	Mathew:	We got the negative $3x$ cause the line was back three. … And we got the plus 6 cause it ran through the 6 on the y-axis. [They graph the equation $y = -3 + 6$.]
45.	Together:	Oh … Oh!!!!
46.	Jack:	Let's take a look at this.
47.	Mathew:	Oh brother! Let's try … let's do this.
48.	Jack:	Positive 6 plus … positive.
49.	Mathew:	Let's try the student he said this 6 x minus 3 [they graph the equation $y = 6x - 3$]. Nope.
50.	Jack:	OK, ah … let's try positive x plus 6.
51.	Mathew:	How about just x plus 6?
52.	Jack:	Positive 3
53.	Mathew:	Three x plus 6?

54.   Jack: Yeah, cause maybe we put it on the wrong side.
55.   Mathew: Yeah, that might do it. [They graph the equation $y = 3x + 6$.]
                  ... Well, almost did it.

Mathew used a_G for m_E (line 44) and $-b_G$ for m_E and a_G for b_E (line 49). They were also using the y-coordinate of b_G for b_E and the x-coordinate of a_G for m_E, alternating between several uses of the x-intercept. They began to notice that the x-intercept changes as m_E changes. They seemed to be moving from focusing on the x-intercept as a reflection of a change in b_E to seeing it as reflection of a change in m_E. In the last part of their discussion of this problem, Mathew and Jack attempted to make a connection between the slope and the x-intercept, as they further explored the effect of changing m_E on a_G. They graphed the equation $y = 3.5x + 6$ and finally settled on a slope of 2.

Excerpt 2 Continued: Mathew and Jack, Problem 1b

63.   Jack: Two ... two ... how about two?
64.   Mathew: No, two will bring it here [uses a pen to show a line through
                  the point $(-1, 0)$].
65.   Jack: No ... we raised it [referring to the slope number].
66.   Mathew: That's right, we did raise it [referring to the slope number] ...
                  So we gotta go lower. Plus 6.
67.   Jack: Plus 6. [They graph the equation $y = 2x + 6$.] ... made it.
68.   Mathew: Yep.
69.   Jack: Why do you have to do 2 just to get 3?

Jack first proposed they use 2 for m_E (line 63). Starting from the line for $y = 3x + 6$, they tried to change the x-intercept by using different numbers for m_E. Because the x-intercept of the line $y = 3.5x + 6$ is to the right of the x-intercept of their previous line, $y = 3x + 6$, and they had wanted to move the x-intercept to the left, they decided to try a smaller number. Finally they used the number 2 for the slope and produced the correct equation. At the end of this excerpt Jack questioned the relation between m_E and a_G.

During these excerpts these two students alternated between several uses of the x-intercept and used specific cases as resources for refining their use of the x-intercept. During their discussions these students used the x-intercept first for b_E and later for m_E, depending on the specific problem context. For the case of $m = 1$ (Problem 1a) using the x-intercept for b_E was productive, whereas for the case where $m = 2$ (Problem 1b) this use of the x-intercept is not applicable. Although these two students initially used a_G for b_E, by the end of the discussion sessions they were no longer using a_G for b_E or only using $-a_G$ for b_E in the case of $m = 1$. This pair refined

their initial use of the x-intercept by narrowing the contexts in which they applied it, coming to use the x-intercept only in the case when $m = 1$.

Mathew and Jack also attempted to make a connection between the x-intercept and the slope, using another conception as a resource. Their attempts to explore how the slope reflects the x-intercept shows that they continued to see the x-intercept as a relevant object, in part because it is a marker for changes in the slope. The last line in this excerpt suggests that at this point they may have moved on to explicitly address the connection between the slope and the x-intercept. Although that was not the case for this pair (Mathew did not respond to Jack's question and they did not return to this topic during the rest of the discussion session), one student in Case Study 2 did explicitly use a procedure to generate slope to recuperate from his initial use of the x-intercept for m_E.

CASE STUDY 2: USING A PROCEDURE

Both of the students in Case Study 2 invoked the use of the x-intercept in their responses to the assessment questions. Mitch used the x-intercept on the pretest five times in the place of b and four times to describe how a line had moved. Although on the pretest David had identified that the x-intercept was not b_E for a problem with slope 1, he had used the x-intercept once for b_E and four times for m_E.

David and Mitch's responses to discussion Problem 1b (Figure 3) were somewhat contradictory. Although they initially generated the correct equation for the line $y = 2x - 6$ rather quickly and agreed that -6 was the correct b_E for this equation, they proceeded to also agree that the correct equation for this line was $y = 2x + 3$ because the line did go through $x = 3$. They discussed these two options for a while (about 30 turns) and then decided to graph the equation $y = 2x + 3$.

Excerpt 3: Mitch and David, Problem 1b

30. Mitch: We can graph it ... [They graph the equation $y = 2x + 3$]. $3x$ ... $3x$.

31. David: This goes by 2, doesn't it [referring to the units in the graph on the computer screen]? 2, 4, 6 [counting on the x-axis].

32. Mitch: This is a 3 right here [pointing to the point $(3, 0)$] ... I ... wait ... let's see ... Isn't y [mumbles] ... I think it would go through $3x$.

38. Mitch: No, see ... look ... $x + 3$ would go through this way [points to the y-intercept for the line for $y = x + 4$]?

39. David: This one is ah, $2x - 6$ ... this one would go through here, but it would be a little more slanted [pointing to the line for $y = x + 4$] ... . So this is wrong! It's been wrong ... . This is not even the line ... . If it was $y = 2x + 3$, it would go through the 3 on the y-axis instead of going through the 3 on the x-axis.

1c. Write an equation for the line graphed below.
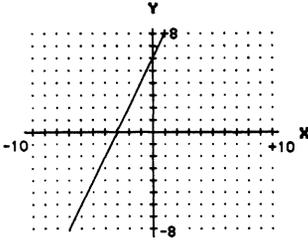
EQUATION:_____

Why?

AFTER GRAPHING

EQUATION:_____

Why?

A student said that the equation for this line was
y=6x-3 because the line goes through the x axis at x=-3
and through the y axis at y=6

Do you think this student was right? | ___YES | ___NO

| ___YES | ___NO

Why or why not?

Why or why not?

Describe below how you could check your answer
without the computer:

FIGURE 4   Problem 1c.

By the end of this excerpt, David had sorted out the difference between the *x* and *y*-intercepts and whether they appear in the equation. During the next problem (see Figure 4) David and Mitch also used the procedure "rise over run" to generate m_E:

Excerpt 4: Mitch and David, Problem 1c

1. Mitch: This 6 [referring to the equation $y = 6x - 3$] would mean that for every one of these there would be 6 of these ... no, first you look

at this, it's minus 3 it's gonna ... [traces the line $y = 6x - 3$ by first finding the y-intercept, then going 1 unit over from the y-intercept and 6 units up]. Right, now if this was $6x - 3$ it would be going that way ... anyway it would be on the bottom portion and ours is in the top section, do you get it?

2. David: So for this one [pointing to the line for $y = 6x - 3$] it would go negative 3 ... 1, 2, 3 ... negative 3 right here, and I would count one up right here, and then you would go 6 up, 1, 2, 3, 4, 5, 6 ... right here right? [Graphs the line on the paper that is not visible.] So the line would go like this right?

This excerpt shows how these two students used a procedure for generating slope to recuperate from the initial use of the x-intercept for m_E. This is an example of how a mathematical procedure can be another type of resource for refining an initial conception.

CASE STUDY 3: USING DESCRIPTIVE LANGUAGE

Students also used descriptive language as a resource for participating in conversations, negotiating meanings (Moschkovich, 1996) and refining initial conceptions. One important aspect of conceptual change documented in the case studies was the refinement of students' verbal descriptions. The analysis of three case studies (6 students) for changes in verbal descriptions showed that the initial descriptions used by five of the six students reflected the use of the x-intercept.[8] This conception was reflected in the description of lines as moving right and left, either instead of or in addition to up and down, as a result of changing b_E. Although some pairs were more successful than others in reaching final agreement on their descriptions, there was some change in each of these six students' use of language to describe the movement of lines.[9] As students progressed through the problems and discussed their initial descriptions explicitly, they stopped referring to horizontal translation and instead focused on vertical translation as the result of changing b_E.

Five of these six students refined their verbal descriptions of the effect of changing m_E or b_E and developed more precise meanings for terms and phrases such

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[8]Initial student descriptions also reflected another conception not discussed in this article, the initial conflating of the role of the two parameters m and b. This conception was reflected in descriptions that did not differentiate between rotation and translation or involved both of these movements simultaneously, and in the initial misunderstandings and extensive negotiations of the meaning of terms such as *steeper, less steep, higher, lower, moves up,* and *moves down*.

[9]Although Monica's descriptions did change, she persisted in describing lines as moving along the x-axis as a result of changing b_E.

as steep, steeper, less steep, moves up–down, moves to the left–right (Moschkovich, 1996). Most of the students came to use descriptive language in ways that are more consistent with the mathematics register[10] (Halliday, 1978; Pimm, 1987). For example, within the mathematics register descriptions of lines refer to the effect of changing b and the effect of changing m as independent. Also, within the mathematics register the effect of changing b in an equation is described as moving a line vertically in the y-axis. Instead, students did not initially separate the effect of changing m and the effect of changing b in their descriptions. Later on they came to refer to these two parameters as independent. The x-intercept was reflected in initial descriptions focusing on horizontal translation. Students' descriptions toward the end of the discussion sessions reflected an increased focus on vertical, rather than horizontal, translation.

I discuss excerpts showing two students who initially used both horizontal and vertical translation to describe the effect of changing b_E. By the end of the discussion sessions, one of the students, Denise, had refined her descriptions to focus principally on vertical translation. Excerpt 5 is taken from the beginning of the discussions sessions while these two students were working on Problem 4b (see Figure 5) and shows how both students initially use horizontal translation in their descriptions.

Excerpt 5: Monica and Denise, Problem 4b

1.	Monica:	[Reading Problem 4b] "If you wanted to … " You would add two to x … would you add two to x? Here's x, right? [traces the y-axis] x … [traces the x-axis].
2.	Denise:	No, remember if you add it moves up [points up from the origin on the y-axis] or to the side [points left of the origin on the x-axis] when you times it, it just stays right in the middle but it moves [rotates fingers counterclockwise and clockwise].
3.	Monica:	[Mumbles.] Right Denise! No, because when you add … it moves to the left or right?
4.	Denise:	Huh?
5.	Monica:	Because when you add it moves to the left.
6.	Denise:	When you add it goes on … moves on the x-axis [points to the negative side of the x-axis].
7.	Monica:	Which axis?
8.	Denise:	Either! … On both! when you add they go on both axes!
9.	Monica:	Because when you add it moves to the sides and parallel.
10.	Denise:	It moves parallel and to the sides.

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[10]A register is "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1978, p. 195). The mathematics register is the set of meanings, words, and structures appropriate to the practice of mathematics.

4b. If you start with the line **y=x (dark)** what would you do to the equation to get the other line (light)?
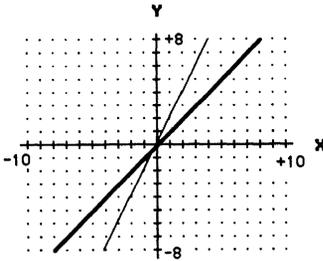
AFTER GRAPHING

A. Add 2 to x
   Why or why not?

____YES   ____NO || ____YES   ____NO
                              Why or why not?

B. Multiply x by 2
   Why or why not?

____YES   ____NO || ____YES   ____NO
                              Why or why not?

C. Both multiply x by 2
   and add 2 to x
   Why or why not?

____YES   ____NO || ____YES   ____NO
                              Why or why not?

FIGURE 5   Problem 4b.

Excerpt 5 is strewn with descriptions reflecting the use of the *x*-intercept as a focus on horizontal translation. Denise used both horizontal and vertical translation in lines 2, 8, and 10 and referred only to horizontal translation in line 6. Monica used only horizontal translation in line 3 and both horizontal and vertical translation in line 9. These descriptions either focused on the line moving "on the *x*-axis" (Denise, line 6) as a result of a change in b_E, or included movement on the *x*-axis well as the *y*-axis, as in "on both axes" (Denise, line 8), and "to the sides" (lines 9 and 10).

Later on, as Monica and Denise explored the result of changing both b_E and m_E, they started to associate a change in b_E with the vertical movement of the line (lines 30–31):

Excerpt 6: Monica and Denise, Problem 4b

23.	Monica:	What if you multiply and add?
24.	Denise:	No, because that would be like over [They are looking at the equation $y = 2x + 2$. She points to $(-2, 0)$]. Yep.
25.	Monica:	It moved up to the left, huh?
26.	Denise:	It moved left one space on the y … on the x-axis.
27.	Monica:	One space?
28.	Denise:	Yeah.
29.	Monica:	To what?
30.	Denise:	What? … wait … It went up 2 on … on y and it went.
31.	Monica:	Oh gosh! … it moved 2 spaces up on the y, aha!
32.	Denise:	And 1 space to the left on x.
33.	Monica:	To the left on x … OK.

During this excerpt, Monica used horizontal translation in her description in line 25. Denise initially used horizontal translation (line 26) but began to focus on vertical translation (line 30), and then returned to a focus on horizontal translation (line 32).

By the end of the discussion sessions, during a problem where the target equation was $y = x - 6$, Denise was referring to the effect of a change in b_E as corresponding primarily to a vertical translation of the line, using the phrase "go down" to describe translation. Monica, on the other hand, continued to describe the result of a change in b_E in terms of movement along the x-axis, using the phrase "move over." In the following excerpt Denise used vertical translation only in line 3, while Monica persisted in referring to horizontal translation:

Excerpt 7: Monica and Denise, Problem 7a

1.	Monica:	[The target equation is $y = x - 6$] Less steep? Yeah.
3.	Denise:	Less steep? No, the same … It will just go down but the line will be the same angle.
4.	Monica:	So it's gonna stay on the y-axis … the x-axis … It's not gonna get any steeper but the line's just gonna move over.

Denise is representative of the majority (five out of six) of the students in the three case studies analyzed in terms of changes in verbal descriptions. By the end of the discussion sessions, five of the six students whose descriptions were analyzed for changes had stopped referring to horizontal translation in their descriptions. This last case study is an example of how descriptive language both reflects conceptual change and can be a resource for refining a conception. Language served

as a resource by providing a vehicle—in addition to gestures and actions—for making a student's focus on the x-intercept explicit and for discussing this conception with their partner.

CONCLUSIONS

The examples presented illustrate four types of resources available to learners for refining conceptions. By using specific cases, students refined their use of the x-intercept for b_E when $m = 1$ from using the x-coordinate of the x-intercept (a_G) as b_E to using the opposite of the x-coordinate of the x-intercept ($-a_G$) as b_E and moved from using the opposite of the x-coordinate of the x-intercept ($-a_G$) for b_E for lines with any slope to using it only when $m = 1$, thus narrowing the contexts in which they applied this conception. Students also used another conception, the slope, to refine their use of the x-intercept, moving from seeing the x-intercept as a reflection of a change in b_E to seeing it as a reflection of a change in m_E and attempting to make a connection between these two conceptions. Students also used a procedure for generating m_E to recuperate from an initial use of the x-intercept as the slope. Lastly, students used descriptive language as a resource for refining their use of the x-intercept, as they moved from focusing on vertical translation to focusing on horizontal translation as a result of changing b_E.

These examples of conceptual refinement point to four types of resources that are representative of the repertoire of resources available to learners for refining conceptions—specific cases, other conceptions, mathematical procedures, and descriptive language. Using specific cases to narrow the problem contexts in which a conception is applied is a productive way to recover from an initial overgeneralization or a conception that was initially applied regardless of the problem context. Making a connection to another conception, not only connects the two conceptions but is also a way to understand the initial conception itself and to develop a more coherent framework in a domain. Lastly, descriptive language can be an important resource for refining a conception by providing another vehicle, in addition to gestures and actions, for making a conception explicit and discussing the conception with someone else.

The results in Moschkovich (1992) summarized here corroborate and extend previous studies (Schoenfeld, Arcavi, & Smith, 1994) and show that student responses involving the use of the x-intercept were not superficial mistakes but reflect an underlying conception. Because some students did not stop using the x-intercept even after participating in the discussion sessions, this is a robust conception. However, because many students did show improvement on these assessments, this conception can be refined. There are several plausible explanations why this conception may be resilient. First, the x-intercept is perceptually salient on the graph and thus, students will expect it to appear somewhere in the

equation as well. Moreover, using the x-intercept is a productive conception in some problem contexts, so students will continue using it because it works. The x-intercept may also be so resilient because it is a maker for changes in m_E. If students do not have a clear understanding of the concept of slope they may continue to focus on the x-intercept because it reflects changes in m_E. Lastly, students may associate the right side of an equation with x and expect the parameters m and b to be related to x in some way.

The resources documented in this article contribute to a view of mathematics learning that goes beyond analyzing misconceptions and to an approach to mathematics instruction that goes beyond correcting student misconceptions. The fact that even a robust misconception was refined and that this refinement occurred even in the absence of direct instruction highlights the productive aspect of students' initial conceptions and the productive resources that learners bring to the classroom. These four types of resources can be used during instruction to support students in understanding mathematical concepts, in moving beyond their initial conceptions, and in recuperating from misconceptions. Teachers can ask students to explicitly consider specific cases as a way to examine initial conjectures. Instruction can also point to other conceptions and mathematical procedures that may help students in refining initial conceptions. Discussing and negotiating descriptions can also support students in refining their descriptive language and using this resource for learning mathematics.

In this study, I extended diSessa's (1993) theory to mathematics by applying a "knowledge in pieces" framework to the analysis of a student conception in mathematics, describing the refinement of a conception in this domain, and documenting the positive resources that students use to refine an initial conception. The first two types resources for conceptual refinement described here, specific cases and connections to other conceptions, parallel the recommendations made by Smith et al. (1993) for describing conceptual change. In addition, in this study I also point to how descriptive language, an aspect of conceptual change that was not explicitly addressed by diSessa's theory, both reflects conceptions and can be a resource for conceptual change. Although diSessa's theory of science learning provided a starting point, a more complete theory of learning mathematics will need to include the role of language in conceptual change.

There are two issues central to a coherent theory of mathematics learning that I did not address in this article that need to be considered in future research. First, a theory of mathematics learning based on diSessa's theory of science learning will need to explore whether and how "phenomenological-primitives" (p-prims), a central aspects of diSessa's theory, may function in mathematics. Future research will need to examine how the construct of p-prims can be extended to mathematics and describe the nature, origin, and evolution of p-prims in different areas of mathematics. Second, a theory of mathematics learning would not be complete without a consideration of how social aspects of learning specifically impact

conceptual change in mathematics, especially the relation between social context and language use.

ACKNOWLEDGMENTS

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