Why Are Some SOLIDS Perfect?

Conjectures and Experiments by Third Graders

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Richard Lehrer and Carmen L. Curtis

In the early grades, the classification of shapes and forms is usually a routine exercise in which children simply identify shapes or memorize taxonomic relationships, for example, that squares are special kinds of rectangles. In contrast, we find that classifying two- and three-dimensional forms can be a fruitful and productive way of learning to think about the properties of these forms and also about the relationships among properties. Along the way, children also learn to participate in an important form of mathematical reasoning—making definitions and exploring the mathematical implications of these definitions.

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This article describes how classification came alive in a third-grade classroom as children searched for "rules," or *properties,* defining the five Platonic solids and created definitions for them. Platonic solids are polyhedra composed of regular polygonal regions with the same number of faces meeting at each vertex. The faces can be bounded by equilateral triangles, as are the tetrahedron, octahedron, and icosahedron; by squares, resulting in the cube; or by regular pentagons, resulting in the dodecahedron. Creating a definition that includes these five solids and only these five solids requires careful consideration of properties and their relationships. Before turning to the classroom activity devoted to Platonic solids, we briefly note some previous experiences that the children had had with shape and form.

Preambles to Investigating Platonic Solids

Thinking about shape and form were part of everyday mathematics for Carmen Curtis and her third graders. As a result, the children had had a wide range of previous experiences with three-dimensional forms before they worked with Platonic solids. For example, they built solids with Polydrons, which are interlocking plastic polygonal pieces; constructed two-dimensional representations, or nets, of unfolded cereal boxes; and examined some properties of the forms that made up each of these representations. Because words like *sides* and *corners* meant different things to different children, the class negotiated common meanings: "An edge is a line where two faces come together."

Students had also previously played a "mystery solid" game using clues to guess the identities of unknown solids. The clues were properties of solids, such as the number of faces or the shape of each face. For example, when told that their teacher was thinking of a solid with triangular faces, the class decided that the mystery solid could be a tetrahedron, a triangular prism, or a square pyramid, which were all solids that they had experienced. Curtis responded with another clue: "Five faces." The students narrowed their focus to the square pyramid and the triangular prism. Curtis gave the third clue: "Three faces come together at each vertex." "Then it *has* to be the triangular prism because the pyramid has four faces meeting at its point." While playing this mystery-solid game, the children talked about which properties were "better clues." For example, the number of faces was popular because it eliminated many possibilities. By analyzing and comparing their descriptions, the students learned that they could figure out the structure of a solid by thinking about its properties. As a follow-up to the game, the children looked for patterns and relationships among the edges, faces, and vertices of various polyhedra. These experiences, which included using definitions and finding patterns among properties of three-dimensional forms, set the stage for further exploration.

Searching for the "Perfect" Solids

To begin a cycle of conjecture and experiments with Platonic solids, Curtis began a lesson by displaying a cube and tetrahedron and informing the class that both were called *perfect* solids (see **fig. 1**). She also presented a square pyramid and a triangular prism and noted that these forms were *not* perfect solids. Jenna looked at these examples and nonexamples and suggested that "maybe all the faces on the solid have to be the same shape." Eric studied the solids displayed in front of the class and, on the basis of previous classroom experiences with faces, edges, and vertices, proposed, "Maybe the number of faces is a multiple of '2,' the number of vertices is a multiple of '4,' and the number of edges is a multiple of '6.' " Jenna continued to examine the two examples while Eric shared his thinking. She suggested, "Maybe three faces come together at each vertex." The class watched Jenna demonstrate that her proposed rule held for the cube and the tetrahedron.

The class then had three possible properties to consider as they continued to think about what is needed to make a perfect solid. Curtis added that only five solids are perfect and that they could all be constructed using Polydrons. Patrick immediately said, "That means that they can't have any curved faces." The students' task for the day was to conduct experiments with the Polydron pieces to see if they could find the remaining three solids. To conduct these experiments, students worked either individually or in groups to generate possible candidates.

Possible "rules," or *properties*, that could apply to all five perfect solids were displayed in front of the class and modified as children's experiments

FIGURE 1

A cube and a tetrahedron were presented as examples of "perfect" solids.

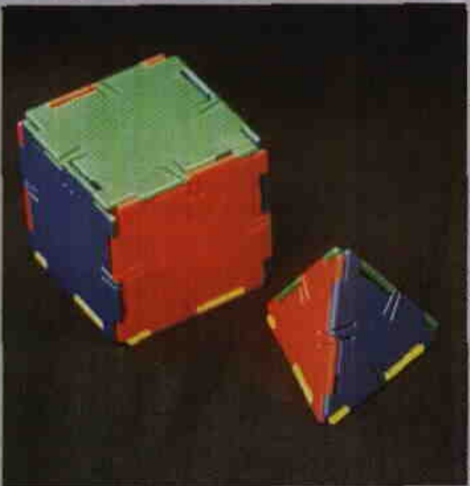


FIGURE 2

The children's first construction

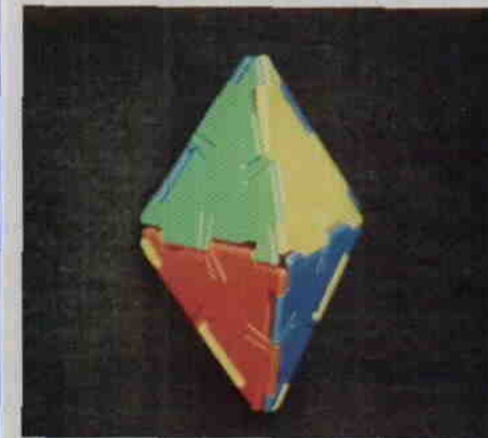


FIGURE 3

The children discovered the octahedron.

were shared with the class. This sharing and discussion of sets of rules was important to help the children learn about constructing and refining definitions. Once students had a solid that they thought could possibly be perfect, they brought it to the attention of the class and justified why they thought it might be perfect. The class discussed whether the suggested rules on the chalkboard applied to this solid. Then Curtis announced whether this solid was, in fact, perfect. Students' conjectures concerning the properties of perfect solids were refined as each different solid was discussed. The students had to decide each time whether they should add something to, or subtract something from, their list of properties or whether one or more of their properties needed to be revised. They tested each new idea against the growing list of examples and nonexamples, all of which were preserved in Polydron constructions so that students could reflect on the evolution of their constructions. It was important to keep a record of both the objects constructed and their properties, so that children had ready, visual access to help develop their definition of perfect solids.

First Candidates

The first solid suggested was constructed from six equilateral triangular pieces. The students who made it noted that it was produced with triangular pieces and looked like two tetrahedrons joined together (see **fig. 2**). The class agreed with those observations but noticed that three faces did not come together at every vertex and that the six faces, five vertices, and nine edges did not match Eric's prediction. Curtis told the class that it was not a perfect solid, which did not help students know *which* of their rules, if any, were correct, so they continued experimenting.

A rectangular prism and a hexagonal prism were then constructed by different groups of children. Both solids had three faces coming together at each vertex, and they followed Eric's pattern: the number of faces was a multiple of 2, the number of vertices was a multiple of 4, and the number of edges was a multiple of 6. But neither solid fit into the perfect group. Again, students had not conducted an experiment that excluded or modified any of their conjectures about the properties of perfect solids, but many were convinced that Jenna's initial idea about all the faces' needing to be identical might be true for perfect solids. This conjecture guided the next cycle of experiment.

Modifying the Rules

The first perfect solid discovered by the class was the octahedron, constructed from eight equilateral triangular pieces (see **fig. 3**). When asked why it might be a perfect solid, the builders said that all its faces were triangular, just like the tetrahedron. After Curtis confirmed that this shape was a perfect solid, she asked the class to think about how that information changed their thinking about the list of possible properties of solids. Many students thought that they could now be almost certain that every face had to be of the same shape, but considering the results of the first experiment and the six triangular pieces, they understood that this property alone would not be sufficient. Eric suggested that his rule about the number of faces, vertices, and edges had to be revised because it did not fit the octahedron. He changed it to say that the number of edges, vertices, and faces all had to be even numbers. Another student pointed out that four faces come together at each vertex of an octahedron, not three, so that Jenna's second conjecture of three faces at each vertex could not be true. One student suggested changing it to "three or four faces come together at each vertex." Another student proposed that "maybe the number just has to be the same at each vertex, but could be any number." All these conjectures were added to the list of possible properties on the chalkboard for further consideration.

Hard on the heels of this discussion, several groups built dodecahedrons constructed from twelve pentagonal pieces (see **fig. 4**). Learning that this solid is perfect did little to change the existing candidates for properties of these solids, although the children were now convinced that the search should be confined to solids with identical faces.

Searching for the Final Perfect Solid

Four of the five perfect solids had been revealed at this point, but the students' task actually became more challenging. The students found several solids that seemed to fit each rule that the class had developed up to that time, but they were told that none of these solids fit into the perfect category. The students began to realize that their set of rules might be incomplete. Each solid that fooled them into thinking that it was the fifth perfect solid had to be analyzed for clues to an additional rule.

Christmas-tree triangles

One pair of students built a solid from four pieces shaped like isosceles triangles (see **fig. 5**), which Patrick named "Christmas tree triangles." All its faces were identical, and three faces came together at each vertex. The class did not have a name for this solid but agreed that it looked like a wedge or a ramp. This solid fit the rules for faces and vertices, but it was not perfect. Why not?

The students were stuck. Curtis asked them to compare the tetrahedron, a perfect solid, with this new wedge-shaped solid. Each was made of four identical triangular pieces, had four vertices, had six edges, and had three faces coming together at each vertex. What was different? Several students noticed that the two shapes were built from different kinds of triangular pieces. The Christmas-tree triangles had two sides that were longer. The class suggested that perhaps the sides of each face all have to be of the same length; they decided that another way to state this requirement was to say that all the edges on the solid must be of the same length.

Diamonds

The class had modified their potential list of properties, but their problem was not yet solved. Several students had been exploring constructions with the rhombus Polydron pieces. These shapes were called *diamonds* by many of the students. A solid constructed from six of the rhombus pieces raised the hopes of many students because they thought that it must be the final perfect solid (see **fig. 6**). When asked to describe this new solid, Patrick

FIGURE 4

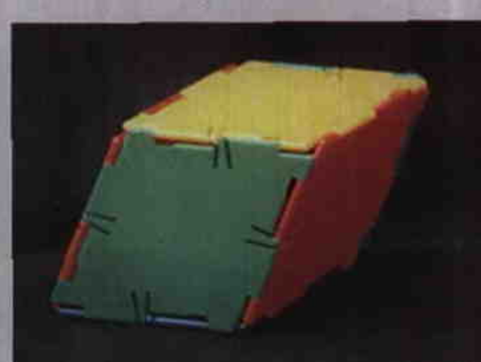
The second perfect solid discovered was the dodecahedron.

FIGURE 5

The Christmas-tree-triangle solid presented an opportunity to think about congruent sides.

FIGURE 6

The "diamond" solid stimulated thinking about congruent angles and the definition of a regular polygon.



said, "It looks like a cube, but the only difference is, it is bent." Another student stated that it looked "like a slanted cube." Patrick pointed out that this slanted solid had six faces just like a cube. Eric had been comparing the cube with the "slanted cube" and observed that "there's three faces coming together to make the same vertex," meaning that three faces met at every vertex on both solids.

Curtis then asked the students to try to discover ways that a rhombus and a square are the same. Harris said, "They have the same equal sides." Several students agreed immediately, but others were confused about this idea that the sides of a rhombus might be equal. To clarify that the sides of the rhombus piece were all of the same length, one student explained and demonstrated that any of the four sides of the rhombus piece could connect to any of the four sides of a square Polydron piece and they would match. The students knew that the square had four sides of equal length, which helped convince them that this fact was also true for the rhombus. They concluded that the "slanted cube" must be a perfect solid because it had faces that were all of the same shape. The sides of each face were all congruent. The same number of faces came together at each vertex on this solid. The same number of faces at each vertex was a generalization that students found preferable to the disjunctive rule of three *or* four faces. The announcement that this solid was not perfect was greeted with disbelief and a little frustration.

The students puzzled over the missing fifth perfect solid. Several said, "We must not have found all the rules [properties] yet!" Ryan, Chris, and Patrick wanted to know why the rhombus piece did not work. They formed two sets of polygons. On the left, they placed four pieces: a pentagon, a square, a small equilateral triangle, and a large equilateral triangle. All these shapes had successfully been used to construct perfect solids. On the right, they placed the isosceles triangle piece and the rhombus piece. Solids made from these shapes did not fit into the category of perfect solids. Why not? The boys moved the isosceles triangle piece aside, since they knew that the shape did not work because its sides were not all equal. That left the rhombus piece. Chris said, "The corners are different!" Curtis asked him what he meant. He laid one rhombus piece on top of another piece to show that two adjacent corners did not match. Ryan said, "The angles have to match," or be congruent. This group tested their thinking on the pentagon, square, and equilateral triangle pieces to show that on the shapes used to build perfect solids, every corner, or angle, matched. They shared their discovery, modifying the rules so that the faces of perfect solids were now shapes with equal angles and equal sides, that is, were regular polygonal regions. See **figure 7** for one student's journal entry.

Finding the last perfect solid

The class decided that the fifth perfect solid would probably be constructed from either square, equilateral triangle, or regular pentagon pieces. The students were fairly sure that the cube was the only solid that could be built using just square pieces unless the squares combined to form rectangular faces. Adam and Harris built a double dodecahedron with twenty-two pentagonal faces, but it did not pass the vertex test. When she was asked directly, Curtis agreed that the fifth perfect solid was made from equilateral triangle pieces. Students explored for a long time. Eric had connected several triangles into a long strip. Curtis held up this item for the class to see and formed it into a ring. She told the students to visualize a solid composed of such a ring with a matching top and bottom. Jenna eventually generated an icosahedron, and several other students followed in quick succession (see **fig. 8**). Some students drew a net on the chalkboard to show one possible arrangement for the twenty pieces so that classmates who were still struggling would have a guide to its construction (see **fig. 9**).

FIGURE 7

A journal entry summarizing a student's exploration of congruent angles

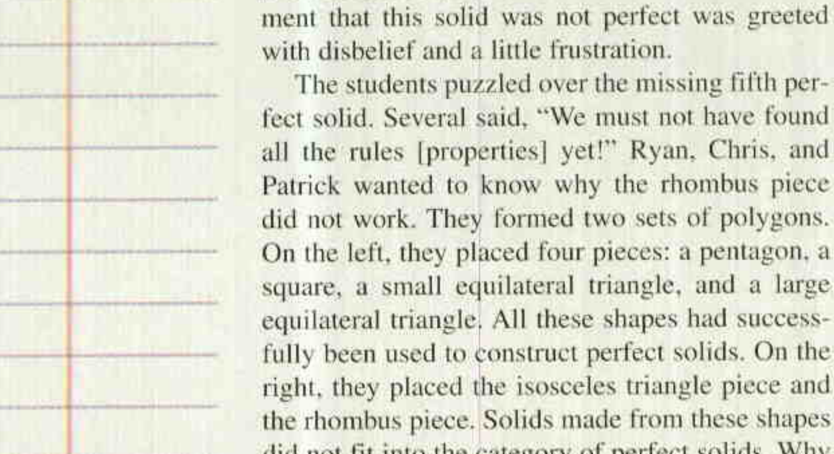
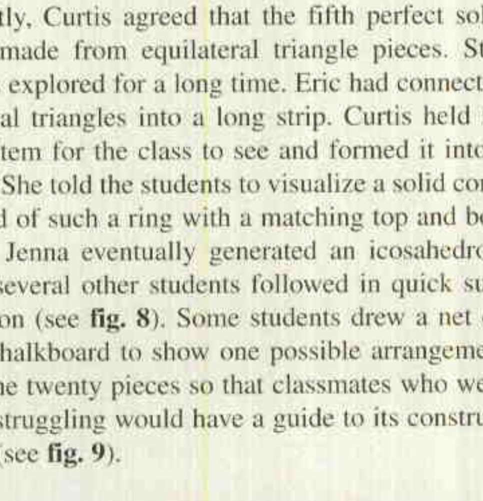


FIGURE 8

Students eventually constructed the icosahedron.



Journals and Reflections

Curtis knew that her students had come away from the morning's investigation with differing amounts of new knowledge and depths of understanding. She was aware of how the class as a whole had progressed in its thinking over the course of the morning, but she needed more information about how *individual* students had progressed in their thinking. She asked each student to summarize in his or her mathematics journal what the class had learned about perfect solids, trying to explain perfect solids in a way that would make sense to their parents or other students in the school who had not investigated them (see **fig. 10**). Students were encouraged to add drawings, diagrams, or nets if those devices would help communicate their ideas more effectively (see **figs. 7** and **9**).

Conclusions

Searching for perfect solids became an avenue for conjecture and experimentation and for thinking about how evidence generated by experimentation could serve to confirm or disprove conjectures. Although we have described a cycle of conjecture and experimentation about perfect solids in the third grade, establishing such cycles for various mathematical topics is important at every grade (Lehrer and Chazan 1998). For example, with older children, one might investigate conjectures about the properties of semiregular, or Archimedean, solids, which are solids with identical vertices but composed of more than one kind of regular polygonal regions; with younger children, conjectures about classes of polygons, such as triangles, could be investigated. Young children often find it surprising that their classmates are willing to consider as triangles figures that are not triangles, and their surprise serves to highlight the need for definitions.

The search for definitions provides an opportunity for a teacher and students to engage in discussion that leads to consensus, or at least to specifying properties in ways that are reproducible and understandable by others. Understanding that properties work together, not in isolation, and that definitions can be modified until the desired mathematical precision is achieved contributes to developing an appreciation of the mathematical virtues of classification (Senechal 1990). Thus, classification is a ready means for children to develop conjectures and explore the implications of their conjectures. In general, mathematical classification is a forum in which children can learn to coordinate theory with evidence, an important and powerful idea with applications across a broad range of subject areas (Kuhn 1996).

FIGURE 9

A journal entry displaying a student's net of an icosahedron

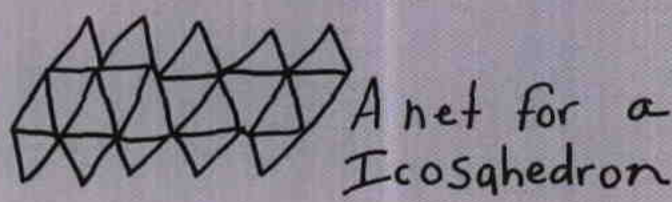


FIGURE 10

Kara reflected on her learning.

Perfect Solids

Kara

Perfect solids have to fit certain rules. The rules are, all the faces have to be flat and on one solid they have to be the same shape. A face is a flat surface on a 3D shape. Some 3D shapes have curved faces, but a perfect one can't have curved faces. A nother rule is, at each vertex the same amont of faces come together on each different solid. A vetex is a pont were faces come together. On each angle has to match each other on the shape that goes into a solid. On each edge of a solid has to be the same length.

There are five different perfect solids in the whole world! Those solids are: Tetrahedron, Octahedron, dodecahedron, and icosahedron. Plus, a cube. They were first decyverd by the greeks. Each perfect solid repesents a different thing.

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